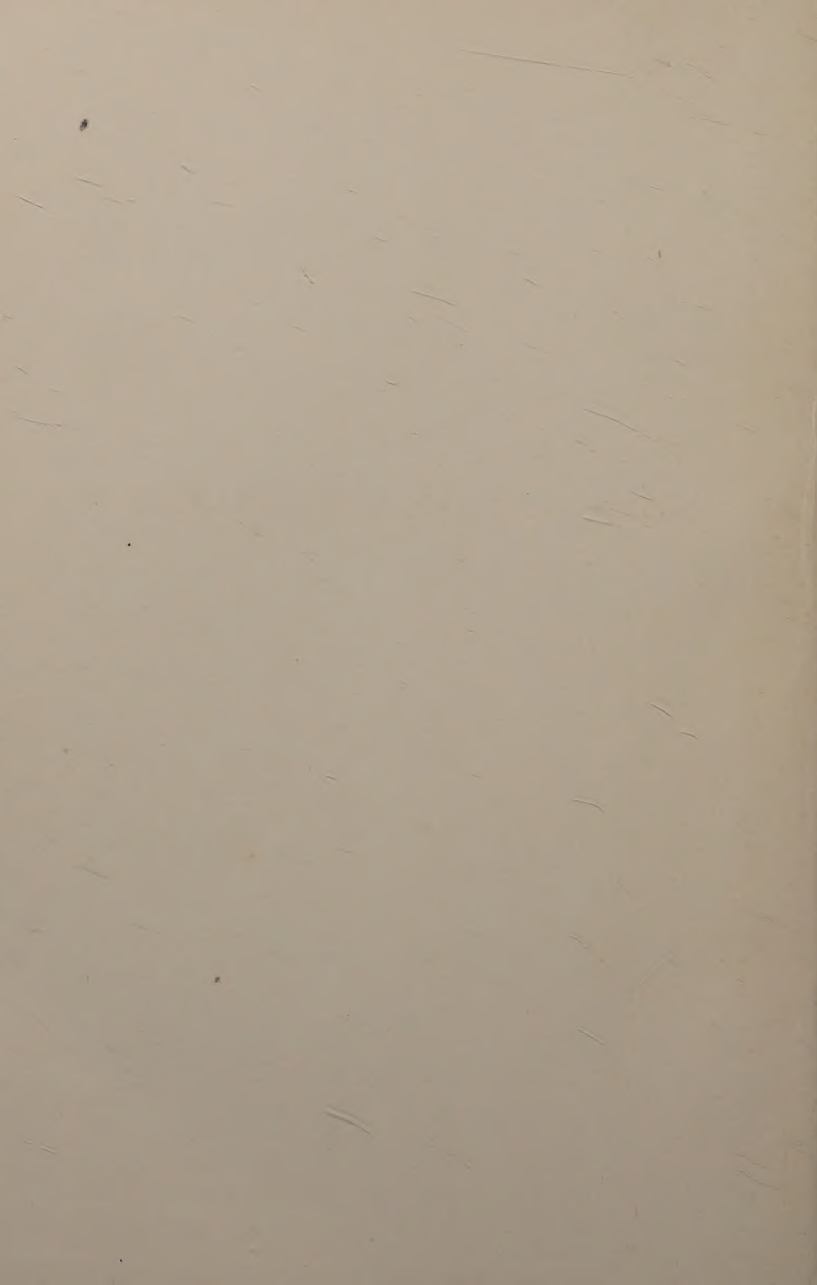


Harold P. Coleman and
Marvin Holmes



AGRICULTURAL MATHEMATICS

FREEHAND LETTERING
FOR
GRAPHS

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M N O P Q R S T U V W X Y Z
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USE CAPITALS FOR TITLES, THUS
GRAPH
FOR
COMPARING PRICES
OF
BUTTER & EGGS

Use small letters for descriptive matter, thus
Price of Butter in Cents per Lb.
or
Period of Ten Years

AGRICULTURAL MATHEMATICS

BY

L. C. PLANT

Professor of Mathematics, Michigan State College

FIRST EDITION
SIXTH IMPRESSION

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PREFACE

A desire to create genuine interest in a required course in mathematics for agricultural students caused the writing of this text. In order to arouse this interest, material has been chosen frequently from agricultural fields in which mathematics has been used as a tool. Experience gained by teaching much of the subject-matter contained in the text, over a period of years, carries the conviction that the student's interest in mathematics is intensified by knowing the source from which the applications came. Because of this fact the author has gone out of his way at times to refer to the particular agricultural bulletin or text from which the problem or data came. As the course progresses the student is impressed with the number of fields of application. The question "Why study mathematics?," is answered.

The chapters on probability and statistics may seem to occupy more space than is justifiable in so brief a course. A knowledge of probability is necessary for a comprehensive understanding of certain agricultural subjects. Since probability is not simple for one who has had a limited amount of mathematics, it seemed desirable to center attention on the fundamentals, but to develop them in detail so that the text might be used for reference when the student has use for probability in later courses in Agriculture.

The student is constantly meeting, in his reading of agricultural reports, averages which have the probable error attached. The chapter on statistics has for its goal the subject of probable error. It was written in detail with the idea in mind that if the student did not take a course in statistics later, he would have

some understanding of standard deviation and probable error and by the use of the text could apply these measures of precision to his statistical data.

Since trigonometry is a prerequisite for college physics and for elementary surveying, and since the allotment of time to agricultural students for the study of trigonometry is short, an attempt has been made to present the fundamentals of the subject briefly without sacrificing clearness. By a careful arrangement of exercises the student is lead to establish certain truths and to derive certain formulas. By this plan the subject can be covered in a minimum number of lessons.

The text has been planned throughout so that the student will learn by doing. A ruler-scale-protractor is placed inside the back cover for the convenience of the student. The long lists of supplementary problems are so varied both in application and in degree of difficulty as to fit the needs of different classes of students.

The large number of applied problems would not have been possible without the help of the men on the staff. Professors L. C. Emmons, S. E. Crowe, and G. G. Specker have been outstanding in their contributions.

If the material in this text, when placed in the hands of others, makes the appeal to agricultural students that it has made in Michigan State College, the author and his staff will be more than repaid for the time spent on its preparation.

L. C. P.

EAST LANSING, MICHIGAN.
December, 1929.

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DIRECTIONS

The material required for the course and the form that the work shall take are as follows:

1. A notebook cover that will hold sheets of letter-size paper, $8\frac{1}{2}$ by 11 inches, and two manila folders for use in submitting work for the examination of the instructor.

2. A scale divided into tenths of inches on one edge, and into millimeters on the other edge.

3. A protractor.

4. A 3H or 4H hexagonal pencil sharpened to a fine point.

5. A good pencil eraser.

6. Graph paper, and computation paper, not ruled. All paper must be letter size.

7. Drawings (including the axes of reference) should be made in pencil, unless the student has had some experience in the use of a drawing pen.

8. All descriptive work upon drawings should be *lettered*, not written and should be *in ink*.

9. All problems and exercises should be done *in ink*.

10. Each drawing must be given a title and enough descriptive matter must be added to make the work clear.

11. The upper left side of each sheet should bear the reference to the pages of the text, and the upper right side of each sheet should bear the name of the student.

12. The scale used in the drawings must be indicated by writing the proper units along the axes and by naming the axes.

13. When two or more curves are drawn upon the same axes, one may be represented by a continuous line, a second by a series of very short broken lines, etc.

14. All work done in the course must be submitted to the instructor at or near the close of the term for final approval.

Note to the student:

Try to have the first draft of your work sufficiently neat in appearance and arrangement to hand in to the instructor.

Look at your result and see if it is a reasonable one. Check when possible.

Your grade for the term will be based upon *accuracy*, *neatness*, and *orderliness*.

AGRICULTURAL MATHEMATICS

CHAPTER I

INTRODUCTION

1. Definitions and Assumptions.—The student has been in the habit of giving a definition, an assumption (axiom), or a theorem as a reason for each step in the demonstration of a theorem in geometry. The following definitions and assumptions are given at the outset in order that the student may follow the same scientific methods in his work in this course:

I. *Addition and Multiplication.*—By means of counting, the student learned to add and to multiply numbers. Addition is abbreviated counting, and multiplication is abbreviated addition.

II. *Subtraction.*—Subtraction is the inverse of addition. To subtract b from a is to find a number r such that if b is added to it, the sum will be a . Thus, to subtract 8 from 12 is to find a number, in this case 4, such that when 8 is added to it the sum is 12. Symbolically, this is written $a - b = r$.

III. *Division.*—Division is the inverse of multiplication. To divide 12 by 4 is to find a number [3], such that when multiplied by 4, the product is 12. Symbolically, this is written $a/b = q$. This is equivalent to $a = bq$, provided that b is not zero. a/b is called a *fraction*.

NOTE.— $a \div b$, a/b , $\frac{a}{b}$, and $a:b$ indicate that a is divided by b .

IV. Zero is defined as $a - a = 0$.

V. The product of zero and any number is zero, and conversely, if the product of two or more numbers is zero, one of the numbers must be zero.

VI. Division by 0 is excluded. Application of this definition to III above for $b = 0$ will justify this statement.

VII. The quotient of $0/a$ is zero when a is different from zero.

VIII. Parentheses preceded by the plus sign (+) may be removed; parentheses preceded by the minus sign (−) may also be removed, if the sign of every term within the parentheses be changed. Parentheses may be introduced in accordance with the same rule. Thus, $3 + 2 + 4 = 3 + (2 + 4)$ and $9 - 4 + 2 = 9 - (4 - 2)$.

IX. The product of two numbers is positive if the signs of the multiplicand and multiplier are alike; negative if they are unlike.

X. The quotient of two numbers is positive if the signs of the dividend and divisor are alike; negative if they are unlike.

XI. *Commutative Laws of Addition and of Multiplication.*

$$3 + 4 = 4 + 3, 3 \times 4 = 4 \times 3.$$

When generalized

$$a + b = b + a, ab = ba.$$

XII. *Associative Laws of Addition and Multiplication.*

$$(3 + 4) + 5 = 3 + (4 + 5), (3 \cdot 4)5 = 3(4 \cdot 5).$$

When generalized

$$(a + b) + c = a + (b + c), (ab)c = a(bc).$$

XIII. *Distributive Law.*

$$3(4 + 5) = 3 \cdot 4 + 3 \cdot 5.$$

When generalized

$$a(b + c) = ab + ac.$$

XIV. If the numerator and the denominator of a fraction are multiplied or divided by the same quantity (not zero) the value of the fraction is not changed. $a/b = am/bm$.

XV. Changing the sign of either the numerator or the denominator of a fraction is equivalent to changing the sign of the fraction.

$$\frac{-a}{b} = -\frac{a}{b}, \quad \frac{a}{-b} = -\frac{a}{b}.$$

XVI. $a^m = aaa \cdots m$ factors, where m is a positive integer. a is called the base and m the exponent of the base, and a^m is read *a exponent m* or *a raised to the mth power*.

$$2^3 = 2 \cdot 2 \cdot 2 \text{ and } a^6 = aaaaaa.$$

XVII. Dividing the numerator and the denominator of a fraction by a *common factor* is called *cancellation*.

Example:

$$\frac{a^2 - b^2}{a^2 + 2ab + b^2} = \frac{(a+b)(a-b)}{(a+b)^2} = \frac{a-b}{a+b}.$$

NOTE.—The student should notice that a *common factor* may be cancelled from numerator and denominator by XIV.

XVIII. To add fractions with common denominators, add their numerators and place the sum over the common denominator.

Example:

$$\frac{h}{m} + \frac{k}{m} = \frac{h+k}{m}.$$

XIX. To multiply two or more fractions, multiply their numerators for a new numerator and their denominators for a new denominator.

Example:

$$\frac{h}{k} \times \frac{l}{m} = \frac{hl}{km}.$$

XX. A fraction is divided by a given quantity if its numerator is divided by that quantity or its denominator is multiplied by that quantity. The student *should always* use the first part of this theorem whenever the numerator is a multiple of the given quantity.

Example:

$$\frac{75}{19} \div 25 = \frac{3}{19}.$$

XXI. In the above illustrations the sign (=) has indicated the equality of two expressions. Such a statement of equality between two algebraic expressions is called an *equation*. The expression on either side of the equality sign is called a *member of the equation*.

XXII. The equality that exists between the members of an equation is not destroyed if

- (a) The same quantity or equal quantities be added to each member.
- (b) The same quantity or equal quantities be subtracted from each member.
- (c) Each member be multiplied by the same quantity.
- (d) Each member be divided by the same quantity (not zero).
- (e) Each member be raised to the same integral power.
- (f) Any positive integral root is extracted from both members, except that if an even root is extracted the double sign $\pm$ must be used.

XXIII. Any quantity may be substituted for its equal in a mathematical expression.

Exercise 1

Give the definition or assumption that justifies each of the transformations in the problems that follow:

NOTE.—The student should review the above definitions and assumptions and apply them until he can work the following problems rapidly and in most cases mentally.

$$1. \frac{-a}{-b} = \frac{a}{b}, \frac{a-b}{m+n} = -\frac{b-a}{m+n}, \frac{ka}{k(m+n)} = \frac{a}{m+n}.$$

$$2. \frac{1}{2}, \frac{3}{4}, \frac{1}{6}, \frac{4}{6} \text{ may be written } \frac{3}{6}, \frac{4}{6}, \frac{1}{6}, \frac{4}{6}.$$

$$3. \frac{a}{b}, \frac{h}{ck}, \frac{m}{cb} \text{ may be written } \frac{ack}{bck}, \frac{bh}{bck}, \frac{km}{bck}.$$

4. Two or more fractions may be reduced to a common denominator by multiplying both numerator and denominator of each fraction by a suitable quantity. (How does one determine a suitable multiplier for each fraction in Problem 3?)

5. What factor would you multiply both numerator and denominator of each of the following fractions by in order to reduce them to a common denominator?

$$(a) \frac{x}{x+y}, \frac{y}{x-y}, \frac{x}{y} \quad (b) \frac{(a^2+b^2)}{a^2}, \frac{a^2-b^2}{b^2}, \frac{k}{a+b}.$$

6. Combine the following fractions after having multiplied the numerator and denominator of each by a suitable factor:

$$(a) \frac{3}{4} + \frac{1}{2} - \frac{5}{12} - \frac{1}{6}.$$

$$(b) \frac{a+b}{x+y} - \frac{a-b}{x-y}.$$

7. Simplify the following complex fraction and give a reason for each step:

$$\frac{\frac{3}{4} + \frac{1}{6}}{\frac{8}{2} - \frac{1}{3}} = \frac{9+2}{18-4} = \frac{11}{14}.$$

NOTE.—In simplifying a complex fraction multiply the numerator and the denominator of the complex fraction by the least common multiple of the denominators of the individual fractions.

8. Find the least common multiple of the denominators of the individual fractions in the following complex fraction and simplify the fraction:

$$\frac{\frac{7}{8} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{2}}.$$

9. Simplify:

$$\frac{\frac{b}{x^2} + \frac{c}{bx}}{\frac{a}{x} - \frac{c}{y}}.$$

10. Simplify:

$$\frac{\frac{a+b}{x+y} + \frac{a}{x}}{\frac{a}{y} + \frac{b}{x+y}}.$$

11. Find the numerical value of the following expressions when $a = 3$, $b = 2$, $c = 6$, and $d = 5$:

$$\frac{3a^2 - 3ac/b}{a + b + c}, \frac{ab^2 - a^2b}{a^2b - ab^2}, \frac{da^6 + db^6}{ca^6 + cb^6}, \frac{ab}{c - ab}.$$

2. **Type Forms of Factoring.**—By actual multiplication show that the following type forms of factoring are true:

I. $ax + ay = a(x + y).$

Examples:

(a) $9x + 9y = 9(x + y).$

(b) $hz + hw = h(z + w).$

(c) $(3 + b)x + (3 + b)y = (3 + b)(x + y).$

(d) $am + an + bm + bn = a(m + n) + b(m + n) = (m + n)(a + b).$

II. $x^2 \pm 2xy + y^2 = (x \pm y)^2.$

Examples:

(a) $x^2 + 8x + 16 = (x + 4)^2.$

(b) $x^2 - 8x + 16 = (x - 4)^2.$

(c) $z^2 + 2hz + h^2 = (z + h)^2.$

(d) $x^4 + 2x^2 + 1 = (x^2 + 1)^2.$

III. $x^2 - y^2 = (x + y)(x - y).$

Examples:

(a) $4x^2 - 9y^2 = (2x + 3y)(2x - 3y).$

(b) $(3a + 4b)^2 - (5x - 4b)^2 = (3a + 4b + 5x - 4b)(3a + 4b - 5x + 4b) = (3a + 5x)(3a + 8b - 5x).$

IV. $x^2 + (a + b)x + ab = (x + a)(x + b).$

Examples:

(a) $x^2 + 11x + 30 = x^2 + (5 + 6)x + 5 \cdot 6 = (x + 5)(x + 6).$

(b) $x^2 + x - 30 = x^2 + (-5 + 6)x + (-5 \cdot 6) = (x - 5)(x + 6).$

(c) Let $b = a$ in the above type form and compare the result with type form II above.

V. $x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2).$

Examples:

$$(a) \ x^3 \pm 8 = (x \pm 2)(x^2 \mp 2x + 4).$$

$$(b) \ 8a^3 - 27b^3 = (2a)^3 - (3b)^3 \\ = (2a - 3b)(4a^2 + 6ab + 9b^2).$$

$$VI. \ acx^2 + (ad + bc)x + bd = (ax + b)(cx + d).$$

Notice that the coefficient of x is separated into two parts whose product ($adbc$) is composed of the factors ac , the coefficient of x^2 , and bd , the constant term.

Examples:

$$(a) \ 15x^2 + 31x + 14$$

$$= 3 \cdot 5x^2 + (2 \cdot 5x + 7 \cdot 3x) + 2 \cdot 7$$

By separating 31 into two parts, 10 and 21, whose product is equal to $15 \cdot 14$.

$$= 5x(3x + 2) + 7(3x + 2)$$

$$= (3x + 2)(5x + 7).$$

These factors can be determined mechanically as follows: $15 \times 14 = 3 \cdot 5 \cdot 2 \cdot 7$. Form two groups of these factors such that the sum of the products of their factors shall equal the coefficient of x , that is: $(2 \cdot 5) + (3 \cdot 7) = 31$. Then rewrite the expression with 31 expressed by two terms as $15x^2 + 10x + 21x + 14 = 5x(3x + 2) + 7(3x + 2) = \text{etc.}$

Exercise 2

1. State the products of the given factors and check your result by actual multiplication.

$$(x \pm 3)^2, (3x + 4)^2, (a + 5)(a - 5), (3a + 5)(3a - 5), 3(2 + 4a) \\ (2 - 4a), (x + 2)(x + 5), (x + 3)(x^2 - 3x + 9), (x - 3)(x^2 + 3x + 9), \\ (3x + 2)(2x + 5), (3x - 2)(2x - 5), (a + \frac{1}{5})(a + \frac{4}{5}), (x - \frac{3}{2})(x - \frac{1}{2}), \\ (2a + 4)(\frac{1}{2}a + 1).$$

2. State which type of factoring applies to each of the following expressions and give the factors:

$$y^2 + (m + n)y + mn, H^2 + 11H + 30, H^2 - H - 30, H^2 + H - 30, \\ 4x^2 - 9y^2, (a + b)^2 - (c - d)^2, (49)^2 - (41)^2, (\frac{3}{4})^2 - (\frac{1}{2})^2, z^2 + 18z \\ + 81.$$

3. Factor $10x^2 + 23x + 12$ by first rewriting the expression with $23x$ expressed as $15x + 8x$ and then grouping the terms. By the mechanical method given under type VI above, $10 \times 12 = 2 \cdot 5 \cdot 2 \cdot 2 \cdot 3$ and $(3 \cdot 5) + (2 \cdot 2 \cdot 2) = 15 + 8 = 23$. This suggests that $23x$ be expressed as $15x + 8x$, etc. In a similar manner, factor the following:

$$10x^2 - 7x - 12, \quad 48x^2 + 146x + 105.$$

Give the prime factors of each of the following expressions and state the type form used in each case:

- | | |
|--|---|
| 4. $x^3 + 3x^2 + 2x$. | 6. $gt - gx - tk^2 + xk^2$ |
| 5. $4a^2b^2 - (a^2 + b^2 - c^2)^2$. | 7. $x^3 + 8y^3$. |
| 8. $9x^2 + 12xy + 4y^2$. | |
| 9. $(a^3 + 8b^3)(a + b) - 6ab(a^2 - 2ab + 4b^2)$. | |
| 10. $(m^2 - n^2)^2 - (m^2 + n^2)^2$. | |
| 11. $(27y^3)^2 - 2(27y^3)(8b^3) + (8b^3)^2$. | 12. $tv - v^2$. |
| 13. $v^3 - t^2v$. | 14. $t^3 - v^3$. |
| 15. $v^2 - 2vt + t^2$. | 16. $3u^2 - 6uv - 9v^2$. |
| 17. $x^4 - 81y^4$. | 18. $144a^2b^2 - 100x^2y^2$. |
| 19. $x^2 + 20x + 19$. | 20. $1 - 6ab + 9a^2b^2$. |
| 21. $8s^2 - 12st - 6su + 9tu$. | 22. $4x^2 + 4xy + y^2 - x^2 + 16x - 64$. |
| 23. $(a + b)^2 - (a - b)^2$. | 24. $4x^2 + 4xy + y^2$. |
| 25. $a^2 + ac + 3ab + 3bc$. | 26. $35a^2 + 17a + 2$. |

3. Solution of an Equation.—To solve an equation in one unknown letter, is to find all those values of the unknown letter that make the two members of the equation equal. Any such value is said to satisfy the equation and is called a *root* of that equation.

The following examples will illustrate some of the methods used in finding the roots of equations in one unknown letter:

Example 1.—Solve the equation

$$2x + 6 = 3 + 4x.$$

If

$$2x + 6 = 3 + 4x.$$

Then

$$2x + 6 - 4x - 6 = 3 + 4x - 4x - 6. \quad \text{Why?}$$

Or

$$-2x = -3$$

And, therefore,

$$x = \frac{3}{2}.$$

Why?

This value of x is a **root** of the equation provided it makes the two members of the equation equal when it is substituted for x .

Check:

$$2 \cdot \frac{3}{2} + 6 = 3 + 4 \cdot \frac{3}{2}.$$

Or

$$3 + 6 = 3 + 6$$

And

$$9 = 9.$$

Therefore,

$x = \frac{3}{2}$ is a root of the given equation.

Example 2.—Justify each transformation in the solution of

$$ax + b = c + dx.$$

If

$$ax + b = c + dx,$$

then

$$ax + b - dx - b = c + dx - dx - b.$$

Whence

$$ax - dx = c - b.$$

and

$$(a - d)x = c - b.$$

Therefore

$$x = \frac{c - b}{a - d}.$$

Example 3.—Justify each transformation in the solution of

$$x^2 - 11x + 30 = 0.$$

If

$$x^2 - 11x + 30 = 0,$$

then

$$(x - 5)(x - 6) = 0.$$

Therefore,

$$x - 5 = 0 \text{ and } x = 5.$$

Similarly,

$$x - 6 = 0 \text{ and } x = 6.$$

Check: Substituting $x = 5$ in the equation we obtain $25 - 55 + 30 = 0$, and we see that the equation is satisfied for $x = 5$. Similarly, we can show that it is satisfied for $x = 6$. $x = 5$ and $x = 6$ are therefore the *roots* of the equation.

Exercise 3

Solve each of the following equations for x and check your work.

1. $2x + 4 = 5 - 3x.$

2. $10 - (6 - 4x) = 3x + (5 - 7x).$

3. $\frac{3x}{4} + \frac{5}{6} = \frac{1}{2} + \frac{x}{12}.$

4. $\frac{3x - 1}{3} = -\frac{x - 4}{6}.$

5. $\frac{x^2 - 9}{x + 4} = x - 5.$

6. $\frac{2x^2 - 5x - 12}{x} = 3x - (x + 4).$

Find the roots of the following equations by first placing all terms in the first member of the equation and then factoring:

7. $x^2 - 20x + 19 = 0.$

8. $x^2 - x = 12.$

9. $4x^2 + 12x + 9 = 0.$

10. $6x^2 + 13x + 6 = 0.$

11. $acx^2 + (ad + bc)x + bd = 0.$ 12. If in Problem 11, $a = 2$, $b = -4$, $c = 5$, and $d = -1$, what are the numerical values of the roots of that equation?

Miscellaneous Problems

1. (a) Reduce $\frac{1}{a - b}$ and $\frac{1}{a + b}$ to equivalent fractions with a common denominator.

(b) What is the sum of these fractions? Check your result by using the arbitrary values $a = 2$ and $b = 1$. Why not check by using the values $a = 1$ and $b = 1$?

2. (a) Reduce $\frac{2x - 4}{x^2 - 7x + 12}$ and $\frac{4}{x - 4}$ to equivalent fractions with a lowest common denominator and show that when the second is subtracted from the first the result is $\frac{-2}{x - 3}.$

3. (a) Express the numerator and the denominator of each of the following fractions in prime factors:

$$\frac{a^2 - b^2}{y^2 + 2ay + a^2}, \quad \frac{2ay + 2a^2}{a - b}.$$

- (b) What is the product of these fractions? Check by letting $a = 2$, $b = 1$, and $y = 1$.

4. (a) Express the numerator and the denominator of each of the following fractions in prime factors:

$$\frac{x^2 + 7x + 12}{x^2 - 5x + 6}, \quad \frac{5ax + 20a}{3x^2 - 2x - 8}.$$

- (b) Find the quotient of the first fraction divided by the second. Check by letting $x = 1$ and $a = 2$.

5. In an examination set for college freshmen, students were asked to state which of the following expressed equalities are true and which are false:

(a) $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}.$

(b) $(a - 2x)^2 = a^2 - 4ax + 4x.$

(c) $\frac{ab + cb}{b} = a + c.$

(d) $\frac{ac + b}{b} = ac + 1.$

(e) $\frac{c}{a + b} = \frac{c}{a} + \frac{c}{b}.$

(f) $\frac{a + ab}{a} = b.$

(g) $\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}.$

(h) $\frac{2a + b}{2c + d} = \frac{a + b}{c + d}.$

Answer this question and correct those statements that are false.

Perform the indicated operations in the following problems, carrying out all work in the simplest manner.

6. $\frac{5}{24} + \frac{3}{8} + \frac{1}{3} + \frac{9}{12}.$

7. $\frac{5}{1 + 2x} - \frac{3}{1 - 2x} + \frac{4 - 13x}{4x^2 - 1}.$

8. $\frac{3a + 4b}{9x + 5} - \frac{9a + 12b}{9x^2 + 32x + 15}.$

9. $\frac{1}{x(x - a)(x - b)} + \frac{1}{a(a - x)(a - b)} + \frac{1}{b(b - x)(b - a)}.$

$$10. \frac{81}{121} \cdot \frac{44}{27} \cdot \frac{108}{45}.$$

$$11. \frac{a-b}{a^3+b^3} \cdot \frac{a^2-ab+b^2}{9a-9b}.$$

$$12. \left(1 - \frac{x-2}{x^2+x-2}\right) \left(\frac{x+2}{x}\right). \quad 13. \left(1 - \frac{24}{25}\right) \left(1 - \frac{57}{64}\right) \div \frac{7}{25}.$$

$$14. \left(1 - \frac{ab}{a^2-ab+b^2}\right) \left(1 - \frac{ab}{a^2+2ab+b^2}\right) \div \frac{a^3-b^3}{a^3+b^3}.$$

$$15. \left(\frac{x^4-y^4}{x^2-y^2} \div \frac{x+y}{x^2-xy}\right) \div \left(\frac{x^2+y^2}{x-y} \div \frac{x+y}{x^2y-y^3}\right).$$

$$16. \frac{\frac{3}{5} + \frac{4}{3}}{\frac{1}{30} - \frac{5}{6}}.$$

$$17. \frac{1 - \frac{2xy}{(x+y)^2}}{1 + \frac{2xy}{(x-y)^2}} \div \frac{\left(1 - \frac{y}{x}\right)^2}{\left(1 + \frac{y}{x}\right)^2}.$$

$$18. \text{ If } m = \frac{1}{(a+1)}, n = \frac{2}{(a+2)}, p = \frac{3}{(a+3)}, \text{ what is the value of } \frac{m}{1-m} + \frac{n}{1-n} + \frac{p}{1-p}?$$

$$19. \frac{2 - \frac{3}{2} + \frac{2}{3}}{5 - \frac{2}{3} + \frac{3}{2}}.$$

$$20. \frac{1 + \frac{1}{x^2} + \frac{1}{x^4}}{1 + \frac{1}{x} + \frac{1}{x^2}}.$$

$$21. \frac{\frac{x}{x-y} + \frac{y}{x+y}}{\frac{x}{x-y} - \frac{y}{x+y}}.$$

$$22. \frac{x^2-4x-77}{x+8} \div \frac{x-11}{56+15x+x^2}.$$

$$23. \frac{81 - (2+x)^2}{x^2-4} \times \frac{7-x}{(x+2)(11+x)}.$$

$$24. \frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1}.$$

$$25. \frac{x+3}{4} - \frac{x-2}{5} + \frac{x-4}{10} - \frac{x+3}{6}.$$

$$26. \frac{a^2-b^2}{bc} - \frac{ac-b^2}{ac} + \frac{ab-c^2}{ab}.$$

Check your result by letting $a = 2$, $b = 3$, and $c = 4$.

Solve the following equations:

$$27. \frac{2x}{3} - \frac{x-2}{2} - \frac{x}{6} = (4-x).$$

$$28. x(x+3) - 4x(x-5) = 3x(5-x) - 16.$$

$$29. 3 - \frac{5-2x}{3} = 4 - \frac{4-7x}{10} + \frac{x+3}{2}.$$

$$30. (x+9)(x+b) = (x-a)^2.$$

Check by letting $a = 2$, and $b = 3$.

Solve the following by first factoring:

$$31. x^2 - 4x - 12 = 0.$$

$$32. x^2 - 5x = 14.$$

$$33. 6x^2 - 7x + 2 = 0.$$

$$34. 48x^2 + 146x + 105 = 0.$$

35. The area of a trapezoid is equal to the product of one-half the sum of the parallel sides by the distance between the parallel sides. If the parallel sides are represented by b and b' and the distance between these sides is h , express the area A in terms of b , b' and h .

If $A = 324$ square feet, $b = 18.6$ feet, and $h = 20.1$ feet, find b' .

36. An open ditch is 18 inches at the top and 14 inches at the bottom and 20 inches deep. What is the area of its cross-section? If the velocity of flow in this ditch is 3 feet per minute and the flow remains at a constant depth of 16 inches, how many gallons will the ditch deliver in 1 minute?

NOTE.—One gallon = 231 cubic inches.

37. In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. If the sides of the triangle are designated by a , b , and c , express the hypotenuse c in terms of a and b . If $b = 8$ and $a = \frac{3}{4}$ this length, find c .

38. In the transmission of farm power a series of pulleys of different sizes are attached to a shaft. By means of belts power is transmitted to different machines and the speed of each of these machines (number of revolutions per minute) may be regulated by varying the size of its respective pulley.

The rules that follow are taken from a handbook on "Farm and Garden," by L. H. Bailey. State these rules by means of formulas by making the substitutions: d_1 = diameter of driver, d_2 = diameter of driven, n_1 = number of revolutions per minute of the driver, and n_2 = number of revolutions per minute of the driven. Then show that from your first formula the other three may be derived.

Rules for determining size and speed of pulleys or gears.

The driving pulley is called the "driver," and the driven pulley the "driven." To determine the *diameter of driver*, the diameter of the driven and its revolutions, and also revolutions of driver, being given:

$$\frac{\text{Diameter of driven} \times \text{revolutions of driven}}{\text{Revolutions of driver}} = \text{diameter of driver}$$

To determine the *diameter of driven*, the revolutions of the driven and diameter and revolutions of the driver being given:

$$\frac{\text{Diameter of driver} \times \text{revolutions of driver}}{\text{Revolutions of driven}} = \text{diameter of driven}$$

To determine the *revolutions of the driver*, the diameter and revolutions of the driven and diameter of the driver being given:

$$\frac{\text{Diameter of driven} \times \text{revolutions of driven}}{\text{Diameter of driver}} = \text{revolutions of driver}$$

To determine the *revolutions of the driven*, the diameter and revolutions of the driver, and diameter of the driven being given:

$$\frac{\text{Diameter of driver} \times \text{revolutions of driver}}{\text{Diameter of driven}} = \text{revolutions of driven}$$

If the number of teeth in gears is used instead of diameter, in these calculations, number of teeth must be substituted wherever diameter occurs.

39. (a) In elementary physics we find the formula $S = at^2/2$, in which S represents the distance a moving body passes through in the time t when it has an acceleration a . Show that $a = 2S/t^2$ and $t = \sqrt{2S/a}$, that is: Solve the given equation for a and for t .

(b) If $S = 64.32$ feet and $t = 2$ seconds, find a .

40. (a) The equation that connects the force F (expressed in pounds) necessary to move a load W (expressed in tons) on an ordinary wagon is $F = kW$ where k takes different values for different road beds. If for a gravel road $k = 50$, find the force necessary to haul 3.125 tons of coal.

(b) This same force would move what sized load of coal on a Macadam road if $k = 25$?

CHAPTER II

GRAPHING DATA

4. Number Pairs.—A city is located by its longitude (distance east or west from a fixed meridian) and by its latitude (distance north or south from the equator).

Draw a horizontal and a vertical line as axes of reference and mark their intersection O . Mark off units on these lines

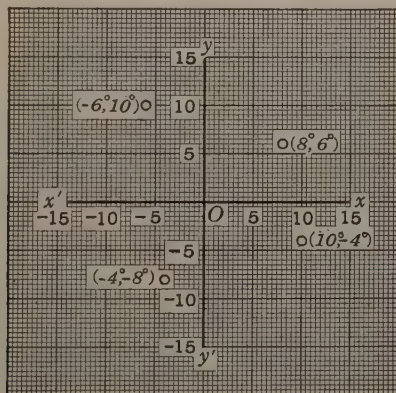


FIG. 1.

to a convenient scale (1 inch = 10°). Give the negative sign to the units along OX' and also to those along OY' .

Locate the cities whose longitudes (abscissas) and latitudes (ordinates) are respectively as follows: $(8^\circ, 6^\circ)$, $(-6^\circ, 10^\circ)$, $(-4^\circ, -8^\circ)$, $(10^\circ, -4^\circ)$, $(20^\circ, -10^\circ)$, $(-15^\circ, 20^\circ)$. Compare your figure with Fig. 1.

5. Rectangular Coordinates.—In Fig. 1 the line XOX' is called the X -axis or the axis of **abscissas**. The line YOY' is called the Y -axis or the axis of **ordinates**. The intersection

of these lines, O , is called the **origin**. The number pair such as $(8^\circ, 6^\circ)$ is referred to as the *coordinates* of the point which it locates, and such a system of locating points in a plane is referred to as a *rectangular system*. The axes divide the plane into four quadrants. The quadrants are referred to as the first, second, third, and fourth. The first quadrant is the one described by XOY , and the others follow in counter-clockwise order.

6. Data.—The data in the table that follows give the price of wheat in dollars per bushel and of flour in dollars per barrel on Nov. 1 for the years 1907 to 1919, inclusive.

	November, 1907	November, 1908	November, 1909	November, 1910	November, 1911
Flour per barrel.	\$3.35	\$4.10	\$5.40	\$4.25	\$4.25

	November, 1912	November, 1913	November, 1914	November, 1915	November, 1916
Flour per barrel.	\$4.65	\$4.15	\$5.75	\$6.81	\$8.50

	November, 1917	November, 1918	November, 1919	
Flour per barrel.	\$10.60	\$10.40	\$10.25	

	November, 1907	November, 1908	November, 1909	November, 1910	November, 1911
Wheat per bushel.	\$0.83	\$1.085	\$1.235	\$0.96	\$0.995

	November, 1912	November, 1913	November, 1914	November, 1915	November, 1916
Wheat per bushel.	\$1.06	\$0.98	\$1.28	\$1.13	\$1.60

	November, 1917	November, 1918	November, 1919	
Wheat per bushel.	\$2.25	\$2.37	\$2.37	

(a) With *years* as the axis of abscissas and *price in dollars per bushel of wheat* as axis of ordinates, the number pairs have been located in Fig. 2 and joined by a broken line.

(b) With *years* as the axis of abscissas and *price in dollars per barrel of flour* as axis of ordinates, the number pairs have been referred to the same axis of abscissas and joined by a

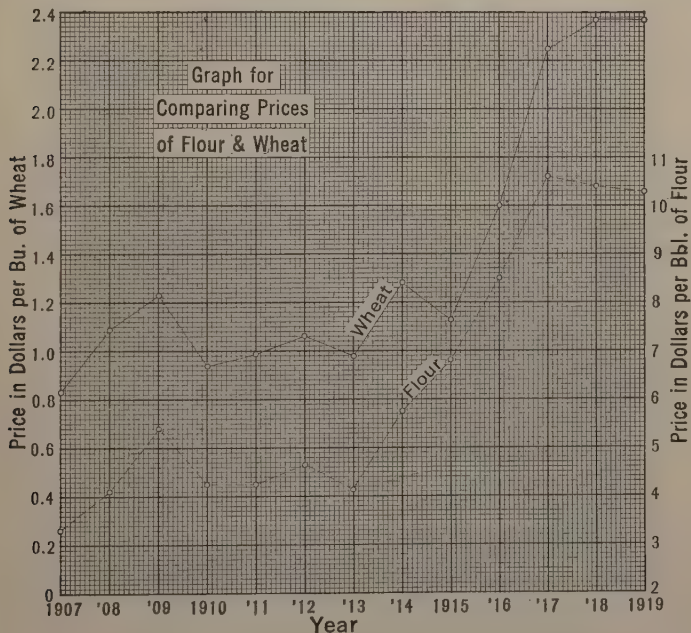


FIG. 2.

dotted broken line. After reading the *Directions*, in front of the text, the student should refer each number pair to this figure, noting the scale used on the axis of ordinates and the manner in which the scale is designated. He should also note the title of the drawing and the way in which the axes are named. He should compare the variation in the prices of

flour and wheat for each year as well as the variation in the price of wheat or of flour during the different years.

Exercise 4

1. Using the graphs of Fig. 2 as a guide, graph the data given in the table. Use a scale on the horizontal axis of $\frac{1}{2}$ inches = 1 year; on the vertical axis, 1 inch = 10 cents.

Year	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917
Butter, cents per pound.....	34	28	32	33	33	32	32	36	35	42	45
Eggs (New York) cents per dozen.....	29	34	38	34	35	40	38	62	44	46	50

These prices are for Nov. 1 from 1907 to 1917, inclusive.

Is there any advantage in using two axes of abscissas (one for price in cents per pound of butter, and the other for price in cents per dozen of eggs) as was done in Fig. 2?

Can one arrive at any conclusions by comparing these graphs?

Should one expect the same relative changes in these graphs as were displayed in Fig. 2?

2. Draw graphs of the data given below.

- Compare prices of potatoes, beans, and apples. Scale $\frac{1}{2}$ inch = 1 year, 1 inch = 1 dollar.
- Compare prices of cotton and wool. Scale: $\frac{1}{2}$ inch = 1 year, 1 inch = 5 cents.
- Compare prices of anthracite and bituminous coal. Scale $\frac{1}{2}$ inch = 1 year, 1 inch = 1 dollar.

NOTE.—Prices given are for Nov. 1 of each year.

Year	Potatoes, per 180 pounds	Beans, per bushel	Apples, per barrel	Cotton, cents per pound	Wool, Ohio and Pennsyl- vania, washed (Boston), cents per pound	Anthra- cite per ton	Bitumi- nous (Pitts.) per ton
1907	\$1.50	\$2.17	\$2.00	11.3	32	\$5.00	\$3.20
1908	2.12	2.40	2.50	9.4	30	5.00	3.25
1909	1.30	2.75	2.30	15.1	35	5.00	3.40
1910	1.25	2.80	3.00	14.6	29	5.00	3.23
1911	2.37	4.80	2.00	9.4	26	5.00	3.15
1912	1.50	4.95	2.00	11.7	28	5.25	3.65
1913	2.10	5.35	2.30	14.1	23	5.25	3.55
1914	1.46	2.27	1.96	6.8	34	5.50	3.65
1915	1.85	2.90	2.21	11.9	42	5.50	3.70
1916	4.40	5.06	2.93	19.6	49	6.00	4.20
1917	4.25	15.50	2.75	28.8	73	6.17	4.10

3. Following are data and graphs that exhibit yields of corn under different treatments of the soil. Examine these data (page 20) and graphs and then repeat the graphical work:

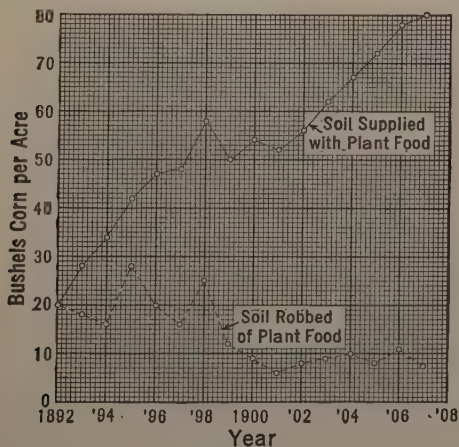


FIG. 3.

The following Table of Data is quoted from *Bulletin 79 Tenn.*

Year	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04	'05	'06	'07
Kind of season	Fair	Fair	Fair	Very good	Good	Fair	Very good	Fair	Fair	Very good	Fair	Good	Very good	Good	Excellent	Very good
Yield in corn in bushels per acre on land on which crop is fed and manure returned to soil....	20	28	34	42	47	48	58	50	54	52	56	62	67	72	78	80
Yield in corn in bushels per acre on land on which the crop is marketed each year.....	20	18	16	28	20	16	25	12	9	6	8	9	10	8	11	7.5

4. The mean temperature readings at the United States Weather Bureau at East Lansing, Michigan, for the month of December, 1913 and 1914, are as follows:

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Temperature in degrees, 1913.....	52	52	40	41	42	38	36	26	28	34	28	42	42	34	32	38
Temperature in degrees, 1914.....	52	43	34	35	36	34	30	32	28	35	20	20	22	8	8	11

Day	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Temperature in degrees, 1913.	35	27	29	28	22	24	28	31	28	12	8	26	29	30	30
Temperature in degrees, 1914.	10	20	25	14	16	12	14	9	2	0	16	26	31	22	11

With days of the month as axis of abscissas and temperature as axis of ordinates, locate the points whose coordinates are given in the table for December, 1913, and draw a broken line through these points, taking them in order. On the same axes and to the same scale carry out the work, using the data for December, 1914. Before selecting the scale for your graphs, note that the range of units on the axis of abscissas is 31 and that the range of units on the axis of ordinates is 52 (this range is the

difference between the highest and the lowest temperature, that is, $52^{\circ} - 0^{\circ}$).

NOTE.—The scale to be used in the graphing of data may depend upon a number of things. One of the most important of these is known as interpolation by means of the graph. An illustration of this particular point will be given later in connection with the use of a graph in finding the square root of a number.

In locating the axes for the temperature curves, is it necessary to provide for the location of points in the second or the third quadrant? Might it be necessary to provide for the location of points in the fourth quadrant? From your graphs determine the number of days in December, 1913, when the temperature was lower than that on the corresponding day in 1914.

5. The mean temperature readings at the U. S. Weather Bureau at East Lansing, Michigan, for the month of December, 1918 and 1919 are as follows. Draw graphs of these temperatures.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Temperature in degrees, 1918.....	24	34	32	29	32	32	40	32	36	32	38	36	40	38	36	38
Temperature in degrees, 1919.....	20	12	12	20	27	27	28	24	26	10	14	35	24	13	8	8

Day	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Temperature in degrees, 1918.	35	30	40	42	47	47	34	39	18	22	21	14	21	28	33
Temperature in degrees, 1919.	8	8	13	14	24	26	26	17	14	32	27	14	20	28	29

6. The data given in the following tables express the percentage of butter fat in the milk of the individual cow for successive milkings for a period of 1 week. As the tables indicate, these milkings were made twice a day in some cases and four times in other cases. The variation in the butter fat during the period of test is best exhibited by means of a graph.

Tests of the milk of cow No. 677 for 7 days give the following results: Cow No. 677. Time expressed in days, four milkings per day.

Number milkings...	1	2	3	4	5	6	7	8	9	10	11	12	13
Percentage butter fat	3.4	5.0	3.0	3.7	3.7	3.7	3.0	3.7	3.2	3.8	3.5	3.2	8.8

Number milkings...	14	15	16	17	18	19	20	21	22	23	24	25	26
Percentage butter fat	3.4	3.0	2.9	3.2	2.9	3.0	3.5	3.1	3.0	2.8	3.4	2.9	4.0

These data may be represented graphically by locating the number pairs with *number of milkings* as axis of abscissas, scale: $\frac{1}{4}$ inch = 1 milking, and *per cent of butter fat* as axis of ordinates: scale 1 inch = 1 per cent. The law in Michigan requires that *legal milk* shall test a minimum of 3 per cent butter fat. Draw a line parallel to the axis of abscissas and cutting the axis of ordinates at 3. How many times does this line cut the curve? Was the milk of this cow up to the legal requirements at all times? What is the arithmetical average¹ of these tests? What is the range of variation of these tests? Does it appear from these data that "the milk of a single cow varies but little in its percentage of butter fat within comparatively short periods"?

Cow No. 855. Range of butter fat, 3.9 per cent.

Number milkings	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat	5.6	4.9	4.2	3.0	3.4	4.0	4.0	3.4	3.6	3.5	3.0	1.7	5.0	4.2

Number milkings	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Percentage butter fat	3.5	4.7	3.6	3.8	3.6	3.0	4.3	3.1	3.3	3.3	4.0	3.5	4.1	3.7

Cow No. 862. Range of butter fat, 3.0 per cent.

Number milkings	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat	2.5	4.4	2.9	2.8	3.2	3.3	3.6	3.5	3.4	2.2	2.8	3.8	3.0	3.7

Number milkings	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Percentage butter fat	2.9	3.2	2.8	2.8	3.4	2.0	2.5	3.8	3.8	4.2	4.0	5.0	3.7	3.5

7. Following is a copy of a data sheet giving the number of pounds of milk, percentage of butter fat, and pounds of butter, for each of 28 milkings for the period of Jan. 8 to 15, 1919, for cow No. 183803, Michigan State College.

¹ See Chap. XIV for definition of arithmetical average.

HOLSTEIN-FRIESIAN ADVANCED REGISTRY OF THE HOLSTEIN-FRIESIAN
 Cow, COLLEGE BELLE BUTTER LASS H. B. No. 183803
 (Certified yield of milk and fat)

	A. M.			A. M.			P. M.		
	Milk, lbs.	Fat, per cent	Fat, lbs.	Milk, lbs.	Fat, per cent	Fat, lbs.	Milk, lbs.	Fat, per cent	Fat, lbs.
Jan. 8									
9	27.4	3.2	0.877	25.2	3.4	0.857	25.2	3.8	0.958
10	25.5	3.3	0.842	24.0	3.9	0.936	24.8	3.6	0.893
11	26.0	3.3	0.858	25.4	3.3	0.838	25.7	5.1	1.311
12	25.6	4.4	1.126	23.5	3.7	0.870	25.5	3.4	0.867
13	25.8	3.0	0.774	26.5	3.3	0.875	24.3	3.2	0.778
14	25.0	3.4	0.850	24.3	3.65	0.887	23.6	3.3	0.780
15	26.7	4.3	1.148	24.5	3.7	0.907	24.9	3.5	0.872

	P. M.			Total	
	Milk, lbs.	Fat, per cent	Fat, lbs.	Milk, lbs.	Fat, lbs.
Jan. 8	25.4	3.85	0.978	25.4	0.978
9	25.8	3.4	0.877	103.4	3.569
10	25.4	3.7	0.940	99.6	3.611
11	25.8	4.2	1.084	102.9	4.091
12	25.9	3.75	0.971	100.5	3.834
13	25.7	4.2	1.079	102.3	3.506
14	26.2	4.5	1.179	99.1	3.696
15				76.1	2.925

Total milk in 7 days..... 709.6
 Total fat in 7 days..... 26.212
 Butter equivalent..... 32.765
 Average per cent fat for 7 days..... 3.694

Using Fig. 2 as a guide, compare by means of graphs:

- (a) The number of pounds of milk in each milking with the corresponding pounds of butter fat.

(b) The number of pounds of milk in each milking with the corresponding percentage of butter fat.

8. The results of tests of the milk of the cows numbered 889, 729, 978, 6, 129, and 1 are given in the following tables. By means of graphs, exhibit the variations in the butter fat.

Cow No. 889. Range of butter fat, 3.8 per cent.

Number of milings.....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat.....	5.3	6.3	4.6	6.8	4.8	4.8	3.6	4.5	4.7	3.4	4.6	4.2	4.0	3.6

Number of milings.....	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Percentage butter fat....	3.7	4.0	3.6	4.5	3.8	3.9	4.0	4.4	3.9	4.0	4.1	4.5	3.5	4.0

Cow No. 729. Range of butter fat, 3.8 per cent.

Number of milings.....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat.....	4.1	3.6	4.2	3.5	3.5	3.4	3.6	4.2	4.6	3.9	5.3	6.3	5.4	4.8

Number of milings.....	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Percentage butter fat.....	4.9	4.0	3.6	3.5	3.7	3.2	2.6	2.7	2.5	3.5	2.9	3.0	3.5	2.0

Cow No. 978. Range of butter fat, 5.0 per cent.

Number of milings.....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat.....	2.2	4.2	6.6	4.4	4.2	1.6	4.5	3.3	4.0	5.6	6.0	5.0	3.7	3.0

Number of milings.....	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Percentage butter fat.....	3.6	3.2	2.9	2.7	2.9	2.7	2.5	2.6	2.5	2.5	2.3	2.5	2.3	2.0

Cow No. 6. Range of butter fat, 6.3 per cent.

Number of milking.....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat.....	5.8	6.2	6.0	7.2	6.0	7.8	2.9	8.4	9.2	6.4	5.7	7.7	6.1	7.0

Cow No. 29. Range of butter fat, 6.6 per cent.

Number of milking.	1	2	3	4	5	6	7	8	9	10	11	12	13
Percentage butter fat	3.4	4.0	3.4	5.0	3.8	4.5	3.5	3.7	6.8	5.1	3.0	5.2	9.6

Cow No. 1. Range of butter fat, 1 per cent.

Number of milking.....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Percentage butter fat.....	4.1	4.5	4.3	4.7	4.3	4.8	4.5	5.1	4.7	4.7	4.5	4.6	4.5	4.8

CHAPTER III

NUMERICAL COMPUTATION

7. Errors of Measurements.—Data that have been obtained by measuring or weighing are only approximations to their values. Because of this fact the results of computations that make use of such data are approximations that may prove very misleading unless both data and result are carefully examined with reference to so-called significant figures.

8. Unit of Measure.—Before defining significant figures, let us analyze the measurement of a length. Suppose that one uses a scale that is divided into inches and tenths of inches with which to measure and that he wishes to obtain data to the nearest tenth of an inch. In using such a scale, one estimates tenths of the smallest division. Suppose the length considered should have a measure of 6.78 inches where the 8 has necessarily been estimated. Since the length is to be read to the nearest tenth of an inch, the measured length would be recorded as 6.8 inches. If the above estimate had been 5 instead of 8 the result would have been recorded as before, 6.8.

Again let us suppose that the length considered should have a measure of 6.84 inches. This result would also be recorded as 6.8 inches. *We shall think of the measured value of 6.8 inches then as any value between 6.75 inches and 6.85 inches.* Such a result is often written as 6.8 ± 0.05 inches. In a similar manner we shall think of 6.8 centimeters as any value between 6.75 centimeters and 6.85 centimeters.

Now, 6.8 centimeters is equivalent to 68 millimeters, which is equivalent to 0.068 meter, and each of these numbers is said to have two significant figures. A little thought on the

above explanation of measurement will convince the student that both 6 and 8 are significant and that 0 as used in 0.068 is not significant, since it was introduced in order that 6.8 centimeters might be expressed in terms of a meter, but in no way did it change the accuracy of the measurement.

9. Significant Figures.—Any of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant. Zero is significant when it is used to show that the quantity in the place in which it stands is nearer to zero than to any other digit. For example, the zero in 6.80 is as significant as the four in 6.84 if the value of the quantity is known to be nearer to zero than to any other digit. As stated above, zeros used to locate the decimal point are not significant. It follows that the number of significant figures in any number is independent of the position of the decimal point. The number 0.00759 has three significant figures, while 204.1 has four significant figures.

Along with such approximations as those that are obtained by measuring or weighing we shall need to use approximations for numbers such as $\pi = 3.141592 \dots$, the square root of irrational numbers such as $\sqrt{2} = 1.41421 \dots$, and logarithms, which will be defined later. In such approximations we shall increase by one the last digit retained¹ in the number if the following digit is 5 or greater, while we shall make no changes in the last digit if it is followed by a digit less than 5. For example, if four significant figures are to be retained as an approximation for π use 3.142, while for three significant figures use 3.14.

Computations Involving Approximate Numbers.—Much of the data that we use in this text have been obtained from departments of agriculture, both national and state. We have accepted these figures as published whether they contained many or few significant figures. In using them we

¹ When the digit cut off is exactly 5, a more scientific rule is to increase the preceding digit by 1 only, when by so doing the last digit retained shall be even.

shall examine our computed results for significant figures in order that these results may not be misleading. We shall illustrate by a few examples.

Example 1.—From the table of Percentage of Plant Foods in various Fertilizing Materials (Art. 29), bat guano is given as containing 1.31 per cent of potash and seaweed is given as containing 2.7 per cent of potash. Let it be required to find the number of pounds of potash in a mixture of 100 pounds of each.

If these percentages were not approximations, our problem would be to add the following numbers:

	Pounds
	1.31
	2.7
Total.....	<u>4.01</u>

If, however, 1.31 means anything from 1.305 to 1.314
and 2.7 means anything from 2.65 to 2.74
then our total might be anything from 3.955 to 4.054

It is clear that the result 4.01 pounds is misleading, as it implies that this result is liable to an error of ± 0.005 , while an analysis of the data shows that it is liable to an error of 0.055. If in the above data we *cut back* the first number to 1.3 and then perform the addition we obtain

$$\begin{array}{r} 1.3 \\ 2.7 \\ \hline 4.0 \end{array}$$

which result implies a possible error of ± 0.05 . We conclude, therefore, that (4.0 ± 0.05) pounds of potash is an approximation to the amount in the mixture.

As a working rule for addition and subtraction of approximate numbers we shall make each number show the same number of decimal places.

Example 2.—In order that we may arrive at a working rule for the retention of significant figures in multiplication and in division of approximate numbers we shall analyze the steps in the following problem: Suppose 46.37 and 5.2 have been obtained by measurement. These numbers have possible errors as indicated:

$$46.37 \pm 0.005$$

$$5.2 \pm 0.05$$

$$\begin{aligned} \text{Product} &= 46.37 \times 5.2 \pm 46.37 \times 0.05 \pm 5.2 \times 0.005 \pm \\ &\hspace{15em} (\text{negligible term}). \\ &= 241.124 \pm 2.3445 \pm 0.0260. \end{aligned}$$

If we cut back the above data to the same number of significant figures in each factor and multiply we have: $46 \times 5.2 = 239.2$. This result is quite as accurate as the first and involves much less labor.

As a working rule for multiplication, we shall make each factor in the product and the product itself show the same *number of significant figures*. In division we shall do likewise with the dividend, divisor, and quotient.

Exercise 5

1. State the number of significant figures in each of the following: 2.36, 2.02, 0.00607, 278.0.

2. Suppose that the following data were obtained in measuring the distance along a straight line from a station *A* to a station *B*; 26.1, 54.3, 8.65, 31.54, 75.102, and 8.364. What changes would you make in these data before proceeding to add the same? If you had been obtaining these data what changes would you have made?

3. In making the measurements for the determination of the area of a rectangle a surveyor obtained the following data: length of rectangle 147.3 feet, width of rectangle 45.68 feet. Determine the area of the rectangle and the possible error in your result. Do you have any criticism to make on the number of significant figures in the factors of this product? How many significant figures will you retain in your computed result?

CHAPTER IV

FUNCTIONS

10. Constants and Variables.—In any expression, certain symbols (letters) may be thought of as taking different values. Those symbols are called *variables*. Symbols which are thought of as not changing are called *constants*. In the equation $y = kx$, x may be thought of as taking different values, while k does not change. Then x is a variable and k is a constant. One variable is a function of another variable when the first depends upon the latter for its value. Thus, in the formula for the circumference of a circle

$$C = 2\pi r,$$

the circumference C is a function of the radius r , since the value of C depends upon the value of r . If $r = 5$, then $C = 10\pi$, while if $r = 7$, $C = 14\pi$. r is called the *independent variable*, and C the *dependent variable*.

Exercise 6

- (a) If 1 pound of sugar costs 9 cents, what is the cost of 35 pounds; of 50 pounds; of 100 pounds; of n pounds?
(b) If 1 pound of sugar costs k cents, what is the cost of n pounds?
(c) If the answer to (b) is given by the equation $C = kn$, where C represents cost, which letters may be considered as variables?
- The area of any circle may be determined by means of the formula $A = \pi r^2$. Which letters in this formula may be considered as variables?
- If r takes the values 1, 2, 3, and 4, what are the corresponding values of A ? The area of a circle is a function of what?
- The number of bushels of wheat a farmer raises is a function of the number of acres he sows. Is it a function of anything else?
- The capacity of a silo is a function of what?

6. In the formula $A = \pi r^2$, A is said to be an explicit function of r , and the equation is said to be solved for A . Using this same function, express r as an explicit function of A , that is, solve the equation for r . If $A = 144$, find r .

7. Express the radius of a circle as an explicit function of its circumference. If the circumference is 12.56, what is its radius?

11. Functional Notation.—In the formula $C = 2\pi r$, the statement that the circumference C is a function of the radius r is expressed in functional notation as $C = f(r)$. In the equation $y = x^2 + 3x - 4$, the fact that y is a function of x may be expressed in functional notation as $y = F(x)$ and is read “ y equals the F function of x ” or “ y equals the F of x ”. The use of functional notation is illustrated by the following exercises:

Exercise 7

1. The number of tons T of ensilage in a silo of diameter 16 feet is expressed as a function of the depth of the ensilage by the formula

$$T = f(d) = 0.051d^2 + 2.5d \text{ where } d \text{ is the depth in feet.}$$

Find T when

$$d = 31 \text{ feet.}$$

Since

$$f(d) = 0.051d^2 + 2.5d,$$

then

$$\begin{aligned} f(31) &= 0.051(31)^2 + 2.5(31) \\ &= 49 + 78 \\ &= 127. \end{aligned}$$

Therefore, T the number of tons ensilage is approximately 127 when the depth is 31 feet.

Note that $f(31)$ is obtained from the $f(d)$ by substituting 31 for d wherever d occurs in the function.

2. If $F(x) = x^2 + 3x - 4$, find $F(2)$. This means that 2 is to be substituted for x wherever x occurs in the function.

Therefore

$$\begin{aligned} F(2) &= 2^2 + 3 \cdot 2 - 4 \\ &= 4 + 6 - 4 \\ &= 6. \end{aligned}$$

Therefore, the value of the function $x^2 + 3x - 4$ is 6 when $x = 2$.

3. If $f(x) = 3x^2 - x - 9$, find $f(3)$, $f(-3)$, $f(0)$, and $f(a)$.
4. If $f(d) = 0.051d^2 + 2.5d$, find $f(40)$, $f(20)$, and $f(40) - f(20)$ and interpret your results after reviewing Problem 1 above.
5. How many tons of ensilage remain in a 16-foot silo which was filled to a depth of 35 feet if 20 feet has been removed from the top?
6. If $f(x) = x^2 + 3x - 1$ and $F(x) = x + 5$, find $f(x)$ when $x = 4$. Also find $f(x) + 3F(x)$ when $x = 2$.
7. The volume V of a cone of revolution is given by the formula $V = \pi r^2 a / 3$. If the radius is 2, express the altitude a as an explicit function of the volume.

8. The area of a triangle is given as a function of its sides by the formula $A = \sqrt{S(S-a)(S-b)(S-c)}$ where S equals one-half the sum of the sides a , b , c (see table of formulas in Appendix).

Find the area of a triangle whose sides are 6.13, 9.27, and 7.06.

12. Variation.—The simplest functional relation between two variables is expressed by the equation

$$y = mx.$$

Many of the problems of the ordinary walks of life may be expressed by functions of this form.

For example: A man works for a given number of dollars per week. The amount that he earns is this given amount multiplied by the number of weeks he works. This is expressed by the function $A = mt$, where t expresses time in weeks.

When the functional relation $y = mx$ exists between two variables y and x where m is a constant, y is said to vary directly as x or is directly proportional to x . m is called the constant of variation or the factor of proportionality.

13. Proportionality.—The idea of proportionality is made evident as follows: In the equation $y = mx$, let x_1 and x_2 (read x -sub-one and x -sub-two, respectively) be any two values of the independent variable x , and y_1 and y_2 corresponding values of the dependent variable y , then substituting corresponding values in $y = mx$ it follows that

$$y_1 = mx_1,$$

and

$$y_2 = mx_2,$$

Therefore

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}.$$

This relation may be expressed in the form of a theorem:
If y varies as x , any two values of y are to each other as the corresponding values of x .

Example: The amount a man earns varies directly as the time he works. If he earns \$392 in 7 weeks, how much will he earn in 10 weeks?

Solution:

$$A = mt.$$

since

$$A = 392 \text{ when } t = 7,$$

Therefore

$$392 = m7$$

and

$$m = \frac{392}{7} = 56.$$

Replacing m by 56, we have

$$A = 56t, \text{ and when } t = 10,$$

$$A = 56 \times 10 = 560.$$

Therefore a man can earn \$560 in 10 weeks.

Let us work this same problem by means of proportion.
Since any two values of the amount A that the man earns are to each other as the corresponding values of the time t that he works, we have

$$\frac{A_1}{A_2} = \frac{t_1}{t_2}.$$

Let $A_2 = 392$. Then $t_2 = 7$ and $t_1 = 10$.

Substituting these values in the formula, we obtain

$$\frac{A_1}{392} = \frac{10}{7}.$$

Therefore

$$A_1 = 560.$$

The following problem illustrates a type of variation which is strikingly different from the type just considered. A man can do a piece of work in m days. How long will it take n men, working at the same rate, to do the work? If T represents the time in days that it takes n men to complete the work, then the question is answered by the equation

$$T = \frac{m}{n}.$$

If it takes a man 10 days to do a piece of work, the time that it takes three men, working at the same rate, to do the work is given by

$$T = \frac{10}{3} = 3\frac{1}{3}, \text{ that is, } 3\frac{1}{3} \text{ days.}$$

The general function representing this type of variation is

$$y = \frac{m}{x},$$

or

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

In this case y is said to vary *inversely* as x or is *inversely proportional* to x .

The last equation is obtained from $y = \frac{m}{x}$ as follows: Substituting the values (x_1, y_1) and (x_2, y_2) in turn in

$$y = \frac{m}{x}$$

we obtain

$$y_1 = \frac{m}{x_1}$$

and

$$y_2 = \frac{m}{x_2},$$

therefore

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

The above problem illustrates one quantity varying inversely as another. *y varies inversely as x when it varies directly as the reciprocal of x.* ($1/x$ is the reciprocal of x .)

We have been considering the dependent variable y as a function of a single independent variable x . The dependent variable may be a function of two or more independent variables. An example of such a function is found in the formula for the lateral area of a right circular cylinder, namely, $A = \pi da$.

The area A is said to vary jointly as the diameter d and the altitude a .

The variable y is said to vary jointly as x and z when it varies directly as the product of x by z.

This variation may be expressed either by the equation

$$y = mxz$$

or

$$\frac{y_1}{y_2} = \frac{x_1 z_1}{x_2 z_2}.$$

The variable y is said to vary directly as x and inversely as z when it varies directly as the quotient of x by z as

$$y = \frac{mx}{z}$$

or

$$\frac{y_1}{y_2} = \frac{x_1 z_2}{x_2 z_1}.$$

Exercise 8

1. If y varies as x and if $x = 4$ when $y = \frac{3}{2}$, find y when $x = 3$.
2. If y is proportional to x and if $x = \frac{3}{4}$ when $y = \frac{3}{8}$, find x when $y = 7$.
3. If y varies directly as x and inversely as z and if $y = 8$ when $x = 12$ and $z = \frac{4}{3}$, find z when $y = 2$ and $x = 6$.

4. If y varies jointly as x and z and if $y = \frac{3}{8}$ when $x = 5$ and $z = \frac{3}{8}$, find z when $y = 3$ and $x = 6$.

5. If y varies as x^2 and if $y = 18$ when $x = 3$, find y when $x = 6$.

6. The volumes of cylindrical silos with equal altitudes are to each other as the squares of their diameters. Compare the volumes of a 14- and a 12-foot silo.

7. The stretch in the spring of an ordinary spring balance is proportional to the weight (force) applied. If a force of 4 pounds stretches it 1 inch, how much will a force of 17 pounds stretch it?

8. How would you graduate (divide) a scale for the spring balance in the above problem?

✓ 9. If the number of acres of grain cut by a machine varies directly as the time in reaping and if 10 acres are cut in 7 hours, how long will it take to cut 7 acres?

10. The area of a circle varies as the square of its diameter. How is the diameter affected when the area is doubled? How is the area affected when the diameter is doubled? Give work showing reasons for your answers.

11. In elementary physics we learned the *Law of Machines*, which is stated as follows: The product of the effort and the distance through which it acts is equal to the product of the resistance and the distance through which it is overcome. In symbols this is expressed as

$$f_1 d_1 = f_2 d_2.$$

Write this formula as a proportion. In turning a jackscrow once around, a force of 50 pounds acts through a distance of 12 feet and an object is raised $\frac{1}{16}$ foot. What was the resistance in pounds?

12. The illumination from a light varies inversely as the square of the distance from the light. Compare the illumination at a distance of 2 feet from the source with that at a distance of 4 feet.

13. If two pulleys are connected with a belt prove that the number of revolutions that they make per minute are to each other inversely as their diameters (see Miscellaneous Problems, Chap. I, No. 38).

14. Composition of Various Foodstuffs.—The three basic nutrients in foodstuffs are protein, carbohydrates, and fats. The figures given in the first three columns of the following table express the percentage of nutrients in these foodstuffs. The figures in the last column give the approximate price of these foodstuffs for the year 1914. In a ton of corn meal

how many pounds of nutrients does one get for the \$36, assuming that the three named nutrients include all the nutrition in the feed?

DIGESTIBLE NUTRIENTS

Feed	Protein, percentage	Carbo- hydrate, percentage	Fat, percent- age	Price per ton
Corn silage.....	1.1	15	0.7	\$ 3.50
Wheat straw.....	0.7	35	0.5	10.00
Alfalfa hay.....	10.6	39	0.9	20.00
Red clover hay.....	7.6	39	1.8	16.00
Oat straw.....	1.0	43	0.9	10.00
Timothy hay.....	3.0	43	1.2	18.00
Wheat bran.....	12.5	42	3.0	30.00
Ground oats.....	9.4	51	4.1	36.00
Corn meal.....	7.5	68	4.6	36.00

15. Carbohydrate Equivalents.—If 1 pound of fat is equivalent in food value to 2.25 pounds of carbohydrates, then fats may be measured in carbohydrate units. The carbohydrate equivalent of any foodstuff, expressed as a percentage, is obtained by multiplying the percentage of fat by 2.25 and adding the product to the percentage of carbohydrate. For example, the carbohydrate equivalent of corn silage, referring to the above table, is $0.7 \text{ per cent} \times 2.25 + 15 \text{ per cent} = 17 \text{ per cent}$, approximately.

16. Relative Values of Feeds.—J. A. Murray, in his "Chemistry of Cattle Feeding and Dairying," states that on the average the relative prices per pound of crude protein and crude fat are about the same, that the price of soluble carbohydrates is relatively less, about two-fifths of the price of the others. This statement is arrived at by taking the market prices (which are necessarily fluctuating) of a large number of foodstuffs and comparing these with their respective digestible

nutrients. He expresses his conclusion in the form of the following equation:

$$S = n[2\frac{1}{2}(P + F) + C]$$

where S represents the price per ton. P , F , and C are the percentages of crude protein, fat, and soluble carbohydrates, respectively; n is a factor of proportionality and varies with the market. It may be determined for any particular foodstuff as soon as the price is given. If in this formula we let $2\frac{1}{2}(P + F) + C = U$ where U stands for *food units*, then $S = nU$ or in the words of variation the value of any foodstuff as based on its food units varies directly as the number of food units in it. It follows by proportion that

$$\frac{S_1}{S_2} = \frac{U_1}{U_2},$$

where S_1 and S_2 represent the values of two different feeds and U_1 and U_2 their respective food units.

Example.—What are the relative values of red clover hay and corn fodder (with ears)?

For red clover hay, $U_1 = 2\frac{1}{2}(7.6 + 1.8) + 39 = 63$, and for corn fodder with ears, $U_2 = 2\frac{1}{2}(3.5 + 1.5) + 47 = 60$. (These values are taken from the table of feeds that follows.) Therefore

$$\frac{S_1}{S_2} = \frac{63}{60} = 1.1 \text{ or } S_1 = 1.1S_2.$$

Our conclusion is that the value of red clover hay as based on food units is slightly greater than the value of fodder with ears.

Exercise 9

1. Determine the carbohydrate equivalent for each of the feeds in the table Art. 14.

2. Using as abscissa, carbohydrate equivalent, and as ordinate, protein, locate a point for each of the feeds in the table Art. 14 and represent their graph by means of a broken line. *Hint:* On horizontal axis, use scale of 1 inch = 10 units; on vertical axis use scale 1 inch = 5 units. You will note from your graph that the carbohydrate equivalents of the

above feeds increase in the order in which they appear in the table. Can you draw a corresponding conclusion concerning the percentage of protein for each of the feeds?

3. If red clover hay costs \$16 per ton, find the corresponding value of corn silage (well matured) by first determining n in Murray's formula. Using the percentages given for the different nutrients in red clover hay and substituting in the formula, we have

$$16 = n[2\frac{1}{2}(7.6 + 1.8) + 39] \text{ or } n = \frac{16}{88}.$$

The relative value of silage is, therefore,

$$S = \frac{16}{88}[2\frac{1}{2}(1.1 + 0.7) + 15] = 5.0.$$

Therefore, if red clover hay is selling at \$16 per ton, the relative value of corn silage is, approximately, \$5 per ton.

4. By means of the formula $\frac{S_1}{S_2} = \frac{U_1}{U_2}$ find the relative values of rutabagas and flat turnips (see table on Analysis of Principal Michigan Feeding Stuffs on page 40).

5. A mature dairy cow should have approximately one part protein to about seven parts of carbohydrates or carbohydrate equivalents.¹ The following feeds give an average ratio of approximately 1 to 7. Verify this statement.

Feed	Protein, percent- age	Carbo- hydrate, percent- age	Fat, percent- age	Cost per ton, Dec- ember, 1914, Lansing
Red clover hay.....	7.6	39	1.8	\$16.00
Clover silage.....	1.3	9.5	0.5	4.50
Corn meal	7.5	68	4.6	36.00
Ground oats.....	9.4	51	4.1	36.00
Barley.....	9.0	66.8	1.6	25.00
Total.....	34.8	234.3	12.6	\$117.50
Approximate mean...	7.0	47	2.5	\$ 23.50

Number pounds protein, carbohydrates, and fats in 1 ton mixture are respectively: 140 pounds, 940 pounds, and 50 pounds.

Three important chemical compounds considered in stock feeding are proteids, carbohydrates, and fats. The percentages of each are given in the following table, taken from "Feeds and Feeding," by W. A. Henry.

¹ Bull. 130, Minnesota Experiment Station.

ANALYSIS OF PRINCIPAL MICHIGAN FEEDING STUFFS

	Total dry matter, pounds	Digestible nutrients			
		Protein, percent- age	Carbo- hydrates, percent- age	Fats, percent- age	Carbo- hydrate equiv- alents, percent- age
Dried Roughage:					
Corn stover.....	59	1.4	31	0.6	32.4
Corn fodder (with ears).....	82	3.5	47	1.5	50.4
Millet.....	86	5.0	46	1.8	50.1
Timothy hay.....	88	3.0	43	1.2	45.7
Red clover hay.....	87	7.6	39	1.8	43.1
Alfalfa hay.....	91	10.6	39	0.9	41.0
Alsike clover hay.....	88	7.9	37	1.1	39.5
Oat and pea hay.....	83	8.3	37	1.5	40.4
Soybean hay.....	91	11.7	39	1.2	41.7
Cow pea hay.....	90	13.1	34	1.0	36.3
Vetch hay.....	87	15.7	37	1.9	41.3
Oat hay.....	88	4.5	38	1.7	41.8
Rye hay.....	92	2.9	41	1.1	43.5
Pea hay.....	89	12.2	40	1.9	44.3
Pea hay (without peas).....	91	7.7	47	0.9	44.0
Wheat straw.....	92	0.7	35	0.5	36.1
Oat straw.....	89	1.0	43	0.9	45.0
Bean straw.....	90	3.6	42	0.7	43.6
Silage and Roots:					
Corn silage (well matured).....	26	1.1	15	0.7	16.6
Mangels.....	9	0.8	6	0.1	6.2
Carrots.....	12	0.9	7	0.2	7.5
Rutabagas.....	11	0.1	8	0.3	8.7
Flat turnips.....	10	1.0	6	0.2	6.5
Potatoes.....	21	1.1	16	0.1	16.2
Concentrates:					
Corn meal.....	90	7.5	68	4.6	78.4
Corn and cob meal.....	90	6.1	64	3.7	72.3
Ground oats.....	89	2.4	51	4.1	60.2
Rye.....	91	9.9	68	1.2	70.7
Wheat.....	90	9.2	68	1.5	71.4
Wheat bran.....	90	12.5	42	3.0	48.8
Wheat middlings.....	89	15.7	53	4.3	62.7
Buckwheat middlings.....	88	24.6	38	6.1	51.7
Soybean meal.....	90	30.7	23	14.4	55.4
Cow pea meal.....	88	19.4	55	1.1	57.5
Pea meal (field).....	89	19.8	54	0.8	55.8
Bean meal (cull).....	87	18.3	54	0.8	55.8
Linseed meal O.P.....	91	30.2	33	6.7	48.1
Cottonseed meal.....	93	37.0	22	8.6	41.4
Dried beet pulp.....	92	4.6	65	0.8	66.8

6. An examination of Murray's formula shows that proteids and fats are valued at two and one-half times as much as carbohydrates. If this is true, what is one paying for a pound of carbohydrates in the mixture of the five feeding stuffs in Problem 5?

7. Determine the value of n based on the mixture in Problem 5 at \$23.50 per ton. (Such a mixture would be subject to less variation than a single feeding stuff.) With this value of n determine the relative value of wheat bran.

Supplementary Problems

1. Numbers and their roots are connected by the formula $N = R^2$. Substitute corresponding values taken from the following table in this formula and see if they check.

Number.....	1	2	3	4	5	6	7	8	9	10
Approximate square root.....	1	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16

Number.....	11	12	13	14	15	16	17	18	19	20
Approximate square root.....	3.32	3.46	3.61	3.74	3.87	4	4.12	4.24	4.36	4.47

Plot these numbers on the horizontal axis (scale: $\frac{1}{2}$ inch = 1 unit) and the roots of these numbers on the vertical axis (scale: 1 inch = 1 unit). Draw a smooth curve through these points.

Use this graph in finding approximately the roots of the numbers 4.5, 6.2, 8.7, 16.3, 18.5. Make a rough check of your work by substituting your results in the formula $N = R^2$.

2. Newspaper clipping, Lansing, Jan. 9, 1918:

The thermometers took a drop of 32° within a few hours yesterday, from 18° above in the afternoon to 14° below at eight o'clock this morning. Following is the record since noon yesterday:

Wednesday	Degrees	Thursday	Degrees
Noon.....	17	Midnight.....	-2
1 p. m.....	18	1 a. m.....	-1
2 p. m.....	18	2 a. m.....	-3
3 p. m.....	17	3 a. m.....	-5
4 p. m.....	14	4 a. m.....	-7
5 p. m.....	11	5 a. m.....	-9
6 p. m.....	8	6 a. m.....	-11
7 p. m.....	-9	7 a. m.....	-13
8 p. m.....	-10	8 a. m.....	-14
9 p. m.....	-9	9 a. m.....	-10
10 p. m.....	-9	10 a. m.....	-1
11 p. m.....	-6	10:30 a. m.....	9

From these data draw a temperature curve.

NOTE.—Draw a smooth curve through located points.

3. An Iowa correspondent writes: "Which is the cheaper for cows, bran at \$36 per ton or oats at \$0.50 per bushel?" (Weight of 1 bushel oats is 32 pounds.)

Editor's reply: "Experiments indicate that oats are slightly superior to bran, pound for pound. Oats at \$0.50 per bushel we regard as superior to bran at \$36 per ton."

The student will justify or disprove these answers by basing his conclusions on digestible nutrients of the foodstuffs.

The Iowa correspondent also asks whether corn at \$0.85 per bushel and linseed meal at \$2.25 per hundredweight is cheaper than bran at \$36 per ton. (Weight of 1 bushel corn is 56 pounds.)

4. Based on digestible nutrients, is a ton of ground oats worth as much as a ton of ground corn?

5. Find the ratio of the value of cottonseed meal to that of linseed meal.

6. Find the ratio of the value of barley and rye. If barley is worth \$30 per ton, what should be the price of rye?

7. Find the ratio of the value of corn silage to that of timothy hay.

8. If the number of acres cut by a machine is proportional to the width of the cutting bar and if a machine with a 5-foot bar cuts 11 acres in 1 day, how many acres can a machine with a $6\frac{1}{2}$ -foot bar cut in a day of equal length?

9. The area of a plat of land varies jointly as the product of the length and breadth. A plat 12 feet wide and 363 feet long contains $\frac{1}{10}$ acre. How many acres in a plat 4 feet wide and 242 feet long?

10. The capacity of a silo varies as the square of its diameter. Compare the capacities of two circular silos of equal heights whose diameters are 14 and 16 feet, respectively.

11. The volumes of two spheres are to each other as the cubes of their radii. The volume of a sphere whose radius is 5 feet is 523.6 cubic feet; what is the volume of a sphere whose radius is 10 feet?

12. The surface of a sphere varies as the square of the radius. The surface is 314.16 when the radius is 5. What is the surface of a sphere whose radius is 9?

13. The mass of a grindstone varies directly as the square of its radius. How much must its radius diminish before the half of its mass is worn away?

CHAPTER V

INDICES AND RADICALS

17. Indices.—We have stated the definition $a^m = a \cdot a \cdot a \cdot \dots$ m factors, when the exponent m is a positive integer.

a is called the *base*. As a direct consequence of this definition we have the following:

LAWS OF INDICES

POWERS WITH POSITIVE INTEGRAL EXPONENTS

1. $a^m \cdot a^n = a^{m+n}.$

Examples:

$$a^4 \cdot a^3 = a^{4+3} = a^7, \quad 2^4 \cdot 2^3 = 2^7,$$

and

$$2 \cdot 2^4 \cdot 2^3 = 2^8.$$

2. $(a^m)^n = a^{mn}.$

Examples:

$$(a^3)^2 = a^{3 \cdot 2} = a^6, \quad (2^3)^2 = 2^6.$$

3. $(ab)^m = a^m b^m.$

Examples:

$$(ab)^3 = a^3 b^3, \quad (2 \cdot 3)^3 = 2^3 \cdot 3^3.$$

4. $\frac{a^m}{a^n} = a^{m-n}$, where m is greater than n .

Examples:

$$\frac{a^4}{a^3} = a^{4-3} = a, \quad \frac{3^4}{3^2} = 3^2.$$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$

Examples:

$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}, \left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}.$$

The definition of a^m given above becomes meaningless if the exponent is negative or fractional. We will now remove the restriction on the exponents and allow them to become negative, fractional, or zero. This necessitates that we lay down the following:

Definitions.—If m and n are positive, then $a^{-m} = \frac{1}{a^m}$

and $a^{\frac{1}{m}} = \sqrt[m]{a}$, which indicates the m th root of a , $a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (\sqrt[m]{a})^n = \sqrt[m]{a^n}$, a^0 is defined as equal to 1, that is, $a^0 = 1$.

An algebraic expression $\sqrt[m]{a}$ or $b\sqrt[m]{a}$ where m is a positive integer is called a *radical expression*. $\sqrt$ is called the *radical sign*, a the *radicand*, and m the *index*.

NOTE.—If a is positive and m an even integer, $\sqrt[m]{-a}$ is said to be an *imaginary quantity*. Such quantities as $\sqrt{-a}$ are written as follows: $\sqrt{-a} = \sqrt{-1} \sqrt{a} = i\sqrt{a}$. The $\sqrt{-1}$ is replaced by i . We shall apply the laws of indices to i as we apply them to any other letter, but in the final result we shall agree to replace i^2 by minus one, that is, $i^2 = -1$.

Example:

$$(-2)^{\frac{1}{2}} \cdot (-3)^{\frac{1}{2}} = i(2)^{\frac{1}{2}} \cdot i(3)^{\frac{1}{2}} = i^2(2 \cdot 3)^{\frac{1}{2}} = -(6)^{\frac{1}{2}} \text{ since } i^2 = -1.$$

Exercise 10

Work the following problems by direct application of the definition of a^m and check your results by applying the laws of indices:

1. $2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2)$ (definition of a^m)
 $= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ (associative law)
 $= 2^5$ (definition of a^m).

Check: $2^3 \cdot 2^2 = 2^5$ (index law 1).

2. $2 \cdot 2^2 \cdot 2^4 =$
3. $4 \cdot 2^3 =$

4. $a^4 \cdot a^2 =$

5. $a^m \cdot a^n =$

$$\begin{aligned}
 6. (3^2)^3 &= 3^2 \cdot 3^2 \cdot 3^2 \text{ (definition of } a^m) \\
 &= (3 \cdot 3)(3 \cdot 3)(3 \cdot 3) \text{ (definition of } a^m) \\
 &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \text{ (associative law)} \\
 &= 3^6 \text{ (definition of } a^m)
 \end{aligned}$$

Check: $(3^2)^3 = 3^6$ (index law 2).

7. $(a^2)^3 =$

8. $(a^m)^n =$

$$\begin{aligned}
 9. (3 \cdot 5)^2 &= (3 \cdot 5)(3 \cdot 5) \text{ (definition of } a^m) \\
 &= 3 \cdot 5 \cdot 3 \cdot 5 \text{ (associative law)} \\
 &= 3 \cdot 3 \cdot 5 \cdot 5 \text{ (commutative law)} \\
 &= 3^2 5^2 \text{ (definition of } a^m).
 \end{aligned}$$

10. $(2 \cdot 4 \cdot 6)^2 =$

11. $(a \cdot b)^2 =$

12. $\frac{3^5}{3^2} =$

13. $\frac{a^5}{a^2} =$

Simplify the following by applying index laws and give reason for each step:

14. $(\frac{5}{3})^2 =$

15. $(\frac{a}{b})^2 =$

16. $a^{\frac{1}{2}} \cdot a^0 \cdot a^{\frac{1}{2}} =$

17. $\frac{3ab^2}{a^2b} =$

18. $(\frac{2^2b^4}{2^4b^2})^{\frac{1}{2}} =$

19. $\frac{75x^2y}{3xy} =$

20. $27^{\frac{1}{3}} =$

21. $16^{\frac{1}{4}} =$

22. $125^{\frac{1}{3}} =$

23. $2^{-3} =$

24. $2^3 =$

25. $49^{\frac{1}{2}} =$

26. $(\frac{3}{4})^{-2} =$

27. $(-27)^{\frac{1}{3}} =$

28. $(2 \cdot 3)^3 =$

29. $(8 \cdot 27)^{\frac{1}{3}} =$

30. $(2 \cdot 3 \cdot 4)^3 =$

31. $(8 \cdot 27 \cdot 64)^{\frac{1}{3}} =$

32. $(8 \cdot 27)^{\frac{2}{3}} =$

33. $(4 \cdot 9 \cdot 16)^{\frac{1}{2}} =$

34. $(3ab^2)^4 =$

35. $(\frac{27a^6b^2}{3a^2b^4})^{\frac{1}{3}} =$

36. $(81a^8b^4)^{\frac{1}{4}} =$

37. $(\frac{2}{3})^3 \cdot (\frac{3}{8})^2 =$

18. Radicals.—When computations involve radical signs, always use fractional exponents in place of the radical sign except in the case of the square root and in some particular cases of the cube root.

The following problems will illustrate the simplification of radical expressions.

Exercise 11

NOTE.—The student should give index law or laws used in each problem.

$$1. \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.$$

$$2. \sqrt{a^3b} = \sqrt{a^2ab} = \sqrt{a^2} \cdot \sqrt{ab} = a\sqrt{ab}.$$

$$3. \sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}.$$

$$4. \sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}, \quad \sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \frac{\sqrt{ab}}{b}.$$

$$5. \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{8}} = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{4} = \frac{9}{4}\sqrt{2}.$$

$$6. \frac{a}{\sqrt{b}} + \frac{c}{\sqrt{4b}} = \frac{a\sqrt{b}}{b} + \frac{c\sqrt{b}}{2b} = \left(\frac{a}{b} + \frac{c}{2b}\right)\sqrt{b} = \frac{2a+c}{2b}\sqrt{b}.$$

$$7. \sqrt[3]{\frac{a}{b}} + 3\sqrt[3]{ab^2} - \frac{1}{\sqrt[3]{27a^2b}} = \frac{\sqrt[3]{ab^2}}{b} + 3\sqrt[3]{ab^2} - \frac{\sqrt[3]{ab^2}}{3ab} = \left(\frac{1}{b} + 3 - \frac{1}{3ab}\right)\sqrt[3]{ab^2}.$$

$$8. \sqrt[3]{\frac{a^{\frac{1}{2}}b^2}{a^4b^{\frac{1}{2}}}} = \left(\frac{a^{\frac{1}{2}}b^2}{a^4b^{\frac{1}{2}}}\right)^{\frac{1}{3}} = \left(\frac{b^{\frac{3}{2}}}{a^{\frac{7}{2}}}\right)^{\frac{1}{3}} = \frac{b^{\frac{1}{2}}}{a^{\frac{7}{6}}}.$$

9. By means of fractional exponents show that

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b,$$

and that

$$\frac{a}{\sqrt{a} + \sqrt{b}} = \frac{a}{\sqrt{a} + \sqrt{b}} \times \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{a(\sqrt{a} - \sqrt{b})}{a - b}.$$

NOTE.—In the second part of Problem 9 we have rationalized the denominator of the fraction.

Write the equivalent of each of the following expressions without the use of the radical sign or negative exponents:

$$10. \sqrt{c^{-4}}.$$

$$11. \sqrt{27a^6}.$$

$$12. \sqrt{\frac{a^{-6}}{b^{-9}}}.$$

$$13. \sqrt{\frac{a}{b^{-3}}}.$$

$$14. \sqrt{4^2 + 5^2}.$$

$$15. \sqrt{a^2 + b^2}.$$

Simplify the following:

$$16. \frac{a^2 - a^{-2}}{a^2 - 1}.$$

$$17. \frac{1 - \frac{1}{x^{-2}}}{x^2 - 1}.$$

$$18. (3a^2b)^3 \cdot (27a^6b^2)^{\frac{1}{3}}$$

$$19. \frac{x^{-2} - y^{-2}}{x^{-2}y^{-2}(x^2 - y^2)}.$$

$$20. \frac{ax^2 + by^2}{x^2y^2} \cdot \frac{x^3y^3}{\frac{a}{x^{-2}} + \frac{b}{y^{-2}}}.$$

$$21. \frac{x^2 + 11x + 30}{5x^{-1} + 1} \cdot (x - 6)^{-1}.$$

$$22. \frac{1 + 2x^{-1} + x^{-2}}{1 - 2x + x^2}.$$

CHAPTER VI

LOGARITHMS

a	b	c
1	$2 = 2^1$	
2	$4 = 2^2$	
3	$8 = 2^3$	
4	$16 = 2^4$	
5	$32 = 2^5$	
6	$64 = 2^6$	
7	$128 = 2^7$	
8	$256 = 2^8$	
9	$512 = 2^9$	

19. Index Laws and Logarithms.—The numbers in column a are said to form an arithmetical progression, and the numbers in column b are said to form a geometrical progression.

If columns a and b are extended, the product or quotient of any two numbers in column b may be found by means of these columns of figures.

Example: To find 8 times 16, add the numbers in column a opposite the 8 and the 16 and look up the number in column b opposite this sum, which is 128. To divide 128 by 16 subtract the number in column a opposite 16 from that opposite 128, which gives 3, and look up the number in column b opposite this difference, which is 8. Reasons for the above are found by applying the index laws $a^m \cdot a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$ to the numbers in column c opposite 8, 16, and 128.

The numbers in column a are said to be the logarithms to the base 2 of the corresponding numbers in column b . For example, the logarithm of 2 to the base 2 is 1. It is written $\log_2 2 = 1$. The logarithm of 4 to the base 2 is 2, or $\log_2 4 = 2$. The logarithm of 32 to base 2 is 5, or $\log_2 32 = 5$. We note by these illustrations that the logarithms of these numbers are those exponents of the base 2 that give the numbers.

Definition.—The *logarithm* of a positive number to any positive base is the exponent of the power to which the base must be raised to produce that number. Symbolically, we

may write $a^x = N$ where the exponent x is the logarithm of the number N with respect to the base a or $\log_a N = x$. This is read, the logarithm of the number N to the base a is x .

Exercise 12

By means of the above definition, find the following logarithms:

1. $\log_2 64 =$ 3. $\log_2 512 =$ 5. $\log_3 81 =$
2. $\log_2 256 =$ 4. $\log_3 9 =$ 6. $\log_5 125 =$

7. Students whose initial letter of surname is included in (a) A to F , inclusive, make out a table corresponding to columns a , b , and c above in which the base of the logarithms is 3.

(b) G to N , inclusive, use base 5.

(c) O to Z , inclusive, use base 10, but extend column a to include 0
-1, -2, -3, and -4.

8. By means of the definition of a logarithm show that column II is equivalent to column I.

I	II
$10^3 = 1000$	$\log_{10} 1000 = 3$
$10^2 = 100$	$\log_{10} 100 = 2$
$10^1 = 10$	$\log_{10} 10 = 1$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^{-1} = 0.1$	$\log_{10} 0.1 = -1$

Base.—There are two systems of logarithms in general use, the *common* system with base 10 and the *natural* system with base $e = 2.71+$. We shall use the *common* system of logarithms in our computations and shall abbreviate the symbol $\log_{10} N$ as $\log N$. We shall understand, therefore, that when the base is omitted, the base 10 is understood.

Suppose we wish to find $\log 294$. From the definition of a logarithm, $\log 1000 = 3$ and $\log 100 = 2$. Therefore, $\log 294 = 2$ plus a decimal, since 294 lies between 1000 and 100. Similarly $\log 84 = 1$ plus a decimal.

Suppose we wish to find $\log 0.026$. We note that $\log 0.01 = -2$ and $\log 0.1 = -1$. Therefore, $\log 0.026 = -2$ plus a decimal.

20. Characteristic and Mantissa.—From the above discussion we see that the logarithm of a number is made up of two parts, a whole number, which is called the *characteristic*, and a decimal part, which is called the *mantissa*.

The characteristic may be found by the following rule:
 + For numbers greater than 1, the characteristic is positive and one less than the number of digits to the left of the decimal point; for numbers less than 1, the characteristic is negative and numerically one greater than the number of ciphers immediately following the decimal point.

Applying this rule in determining the characteristics of the following, we have:

$$\log 764.5 = 2 \text{ plus a decimal (the mantissa).}$$

$$\log 76.45 = 1 \text{ plus the same decimal.}$$

$$\log 7.645 = 0 \text{ plus the same decimal.}$$

$$\log 0.7645 = -1 \text{ plus the same decimal.}$$

$$\log 0.07645 = -2 \text{ plus the same decimal.}$$

For convenience in calculating, a logarithm should be written so that the part on the left of the decimal point is positive. The mantissa is always positive. Thus:

$$\log 0.7645 = 9. \dots -10.$$

and

$$\log 0.07645 = 8. \dots -10.$$

The mantissa of a logarithm is obtained from the table of logarithms. This table (II appendix) gives the mantissas of logarithms from 1 to 100. The decimal point precedes each mantissa, and the characteristic of the number is to be supplied by the computer.

21. Fundamental Theorems of Logarithms.

$$1. \log_a MN = \log_a M + \log_a N.$$

$$2. \log_a \frac{M}{N} = \log_a M - \log_a N.$$

$$3. \log_a N^x = x \log_a N.$$

In these theorems M and N are positive numbers, x may be positive or negative, integral or fractional, and the base of the logarithm may be any positive real quantity except unity.

The theorems expressed in words are:

1. The logarithm of the product of positive numbers is equal to the sum of the logarithms of the factors.

2. The logarithm of a fraction (quotient) of positive numbers is equal to the logarithm of the numerator minus the logarithm of the denominator.

3. The logarithm of the power (positive or negative, integral or fractional) of a positive number is equal to the logarithm of the number multiplied by the power.

These theorems may be proved by use of the laws of exponents and the definition of a logarithm, that is

$$\log_a M = x,$$

or

$$a^x = M.$$

THEOREM 1.

$$\text{Log}_a MN = \log_a M + \log_a N.$$

Proof: Let

$$\log_a M = x \text{ or } a^x = M$$

and

$$\log_a N = y \text{ or } a^y = N.$$

Then

$$MN = a^x \cdot a^y = a^{x+y} \text{ (index law 1).}$$

Since the logarithm of the number MN is the exponent $x + y$, we have

$$\begin{aligned} \log_a MN &= x + y. \\ &= \log_a M + \log_a N \text{ (by substitution).} \end{aligned}$$

The student may prove this theorem for three factors.

THEOREM 2.

$$\text{Log}_a \frac{M}{N} = \log_a M - \log_a N.$$

Proof: Let

$$\log_a M = x \text{ or } a^x = M$$

and

$$\log_a N = y \text{ or } a^y = N.$$

Then

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y} \text{ (index law 4).}$$

Therefore

$$\log_a \frac{M}{N} = x - y = \log_a M - \log_a N \text{ (definition of logarithm).}$$

THEOREM 3.

$$\text{Log}_a N^x = x \log_a N$$

Proof: Let

$$\log_a N = y \text{ or } a^y = N$$

Then

$$N^x = a^{xy} \text{ (index law 2).}$$

and

$$\log_a N^x = xy \text{ (definition of a logarithm).}$$

$$\therefore \log_a N^x = x \log_a N \text{ (by substitution).}$$

USE OF TABLE OF LOGARITHMS¹

22. How to Find the Logarithm of a Number.—Find log 764. The characteristic by the above rule is 2. To obtain the mantissa from the table, look in the column headed *N* for 76. Opposite 76 and in the column headed 4 you will find the number 0.8831 which is the mantissa. Therefore, log 764 = 2.8831.

Find log 7647. The mantissa of this logarithm is not given in the table. If we assume, however, that a small change in the number causes a proportional change in the logarithm, then we may proceed by interpolation as follows:

Mantissa of log 765 is 0.8837

Mantissa of log 764 is 0.8831.

¹ See Table II in the Appendix.

We observe that a difference of 1 in the number makes a difference of 0.0006 in the logarithm, or a difference of 0.7 makes a difference of $0.0006 \times 0.7 = 0.00042$ or cutting the number back one place, our correction is 0.0004. The logarithm of $7647 = 3.8831 + 0.0004 = 3.8835$. Much time will be saved in interpolating by considering the mantissas for the moment as whole numbers. Thus, instead of writing 0.0006 write 6, etc.

NOTE.—In cutting back a number, if the digit omitted is 5 or more, increase the preceding digit by 1, as directed in Art. 9.

23. How to Find the Number Corresponding to a Given Logarithm.—Given $\log N = 3.8835$. To find the number N . Since this problem is the converse of the preceding one, we may trace that problem back. We cannot find the mantissa 0.8835 in the table, but we find 0.8837 and 0.8831, and so forth.

THEOREM.—The common logarithm of numbers which differ only by the location of the decimal point have the same mantissa, since the position of the decimal point may be changed by multiplying by powers of 10. The common logarithms of numbers which differ only by multiples of 10 have the same mantissas. Hence the mantissa of 2 is the same as that of 20.

Exercise 13

Problems for practice in finding the logarithm of numbers by use of table.

- | | | |
|--------------------------------|------------------------|----------------------|
| 1. $\log 369.0 = 2.5670$. | 2. $\log 3.990 =$ | 3. $\log 0.9870 =$ |
| 4. $\log 587.0 = 7.8 \times 4$ | 5. $\log 5870000000 =$ | |
| 6. $\log 0.00000587 =$ | 7. $\log 468.4 =$ | 8. $\log 0.004684 =$ |
| 9. $\log 0.9867 =$ | 10. $\log 8467000. =$ | 11. $\log 40,000 =$ |
| 12. $\log 0.9658 =$ | | |

In the following, find the number N whose logarithm is given:

- | | |
|------------------------------|------------------------------|
| 13. $\log N = 3.6599$. | 14. $\log N = 1.3181$. |
| 15. $\log N = 2.0000$. | 16. $\log N = 9.9890 - 10$. |
| 17. $\log N = 7.7396 - 10$. | 18. $\log N = 0.9365$. |

19. $\log N = 1.3168.$

20. $\log N = 6.8145.$

21. $\log N = 3.6990.$

22. $\log N = 8.7351.$

23. $\log N = 9.9804 - 10.$

By use of the fundamental theorems of logarithms and the assumption that if $a = b$, then $\log a = \log b$ (provided a and b are both positive) solve the following problems:

24. $9.20 \times 6.70 =$

Solution.—Let $N = 9.20 \times 6.70.$

Then

$$\begin{aligned}\log N &= \log 9.20 + \log 6.70 \text{ (fundamental theorem and} \\ &\quad \text{the above assumption)} \\ &= 0.9638 + 0.8261 \\ &= 1.7899,\end{aligned}$$

and

$$N = 61.64 \text{ (The student should check result.)}$$

25. $125.0 \times (-8.340) =$

Since the product of a positive quantity by a negative quantity is negative, let

$$N = 125.0 \times 8.340$$

and

$$\begin{aligned}\log N &= \log 125.0 + \log 8.340 \\ &= 2.0969 + 0.9212 \\ &= 3.0181\end{aligned}$$

and

$$N = 1043.$$

The required product is the negative of this result or $-1043.$

26. $\frac{29.84}{264.3} =$

Solution.—Let $N = \frac{29.84}{264.3}$

then

$$\begin{aligned}\log N &= \log 29.84 - \log 264.3 \\ &= 1.4748 - 2.4221 \\ &= 11.4748 - 10 \text{ (adding and subtracting 10)} \\ &\quad - 2.4221 \text{ (Why?)} \\ &= 9.0527 - 10\end{aligned}$$

and

$$N = 0.1129.$$

$$27. \sqrt[3]{0.8124} =$$

Solution.—Let $N = \sqrt[3]{0.8124}$

then

$$\begin{aligned}\log N &= \frac{1}{3} \log 0.8124 \\ &= \frac{1}{3} (9.9098 - 10) \\ &= \frac{1}{3} (29.9098 - 30) \text{ (in order to divide by 3)} \\ &= 9.9699 - 10\end{aligned}$$

and

$$N = 0.9330.$$

$$28. 28.16 \times 0.8415 =$$

$$29. -0.003651 \times 364.0 =$$

$$30. \frac{82.65}{436.0} =$$

$$31. (-2.568)^2(0.01342) =$$

$$32. \text{Show that } (0.02910)^3 = 2.464 \times 10^{-5}$$

$$33. \text{Show that (a) } \sqrt{2.718} = 1.649; \text{ (b) } \sqrt{27.18} = 5.214;$$

$$(c) \sqrt{0.002718} = 0.05214.$$

$$34. \text{Show that (a) } \sqrt[3]{27.18} = 3.007; \text{ (b) } \sqrt[3]{271800} = 64.77;$$

$$(c) \sqrt[3]{0.00002718} = 0.03007.$$

$$35. \text{Show that the square root of } \frac{1}{2} \text{ is approximately } 0.707.$$

$$36. \text{Show that } (\frac{1}{2})^{\frac{1}{3}} = 0.794, \text{ approximately.}$$

$$37. \text{Show that } (0.0004590)^{\frac{1}{3}} = 7.714 \times 10^{-2}.$$

$$38. \sqrt{64.25} \times 0.002163 =$$

$$39. \text{Given that } c = \pi d. \text{ If } c = 296.4 \text{ and } \pi = 3.142, \text{ find } d.$$

$$40. \text{If } A = 4\pi r^2, \text{ find } r \text{ when } A = 957.6.$$

$$41. \text{Show that } \sqrt[3]{(0.6148)^2} = (0.6148)^{\frac{2}{3}} = 0.7232.$$

$$42. \text{Show that } \frac{1}{\pi} = 0.3183.$$

43. The area of a triangle expressed in acres is determined by the formula $A = \frac{\sqrt{S(S-a)(S-b)(S-c)}}{160}$ when the lengths of the sides

are known and expressed in rods. Find A when $a = 165.2$ rods, $b = 201.6$ rods, and $c = 249.4$ rods (see table of formulas in Appendix).

44. If c is the length of the hypotenuse of a right-angled triangle, a and b the lengths of the other two sides, then

$$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}. \text{ Given that } c = 62.43 \text{ and } b = 34.27, \text{ find } a.$$

$$45. \text{Show that } \sqrt{\frac{8.430 \times 0.06413}{0.004795}} = 10.62.$$

24. Exponential Equations.—Equations in which the unknown letter occurs in an exponent is called an exponential

equation. Following is an example of an exponential equation and a method of solution.

Example: If

$$a^x = b, \text{ find } x.$$

Taking the logarithm of both members of the equation

$$\log a^x = \log b$$

and

$$x \log a = \log b.$$

Therefore

$$x = \frac{\log b}{\log a}.$$

In particular, suppose $a = 1.040$ and $b = 1.480$, then

$$x = \frac{\log 1.480}{\log 1.040} = \frac{0.1703}{0.0170} = 10.00.$$

Exercise 14

1. If $3^x = 45$, find x .
2. If $5^y = 462$, find y .
3. If $10^x = 950$, find x .
4. If $10^x = 1000$, find x .
5. If $5^y = 625$, find y .
6. If $3^x = 81$, find x .

25. Double Interpolation with Practical Application.—

Following is a section of a table given in U. S. Weather Bureau *Bulletin* 235.

RELATIVE HUMIDITY, PER CENT-FAHRENHEIT TEMPERATURES
(Pressure = 29.0 inches)

Air temperature t	Depression of wet-bulb thermometer ($t - t'$)																			
	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
40	96	92	88	84	80	76	72	68	64	61	57	53	49	46	42	38	35	31	27	23
41	96	92	88	84	80	77	73	69	65	62	58	54	50	47	43	40	36	33	29	26
42	96	92	88	85	81	77	73	70	66	62	59	55	51	48	45	41	38	34	31	28
43	96	92	88	85	81	78	74	70	67	63	60	56	52	49	46	43	39	36	32	29
44	96	93	89	85	82	78	74	71	68	64	61	57	54	51	47	44	40	37	34	31
45	96	93	89	86	82	79	75	71	68	65	61	58	55	52	48	45	42	39	36	33

*Example:*¹ Air temperature $t = 42.7$

Reading of wet-bulb thermometer $t' = 34.5$

Depression of wet-bulb thermometer $(t - t') = 8.2$

Determine the relative humidity from above table.

Tabulated values are:	8.0	8.5
42	41	38
43	43	39

To a difference of 1° in air temperature (43–42) corresponds a difference of 2 per cent in the relative humidity (43–41) when depression is 8.0° . Therefore, a change of 0.7° in air temperature (42.7–42) causes a change of 0.7×2 per cent = 1.4 per cent in relative humidity. Similarly, for a depression 8.5° , a difference of 1° in air temperature causes a difference of 1 per cent in relative humidity and, therefore, a change of 0.7° causes a change of 0.7 per cent in the relative humidity. Our table may now be enlarged to:

	8.0	8.5
42	41	38
42.7	42.4	38.7
43	43	39

We now see by referring to our enlarged table that when the temperature is 42.7° and the depression is 8.0° the relative humidity is 42.4 and when the depression is 8.5° the relative humidity is 38.7, therefore, by the usual method of interpolation a difference of 0.5° depression causes a difference of 3.7° in the relative humidity (42.4–38.7). A difference of 0.2° depression causes a difference of 1.5 per cent in relative humidity and therefore the required relative humidity is (42.4 per cent – 1.5 per cent) = 40.9 per cent.

¹ In this and following problems we assume that the barometric pressure is 29.0 inches.

Exercise 15

1. Work the above illustrative problem by changing the order of the method, that is, interpolate for temperature 42° and depression 8.2 , which lies between 8.0 and 8.5 . Then do likewise for temperature 43° , and so forth.
2. If air temperature, $t = 44.3$
 Reading of wet-bulb thermometer $t' = 34.5$
 Depression of wet-bulb thermometer $(t - t') = 9.8$
 Determine the percentage of humidity.
3. If air temperature, $t' = 41.8$:
 Depression of wet-bulb thermometer $(t - t') = 2.4$
 Determine the percentage of humidity.

Supplementary Problems

The movement of moisture into a soil from an illimitable supply is controlled by a rate law which may be expressed by the equation $y^n = kt$ where y equals distance in inches through which the movement has taken place, t is the time in minutes, and k and n are constants dependent upon particular soil and solution.

1. In the case of Gila River alluvial soil, $n = 1.86$ and $k = 1.05$. Find the time required for the water to move through 4 inches.
2. In the case of the movement of distilled water into Pennsylvania loam soil, $n = 2.25$, $k = 1.13$, if y is expressed in centimeters. Find the distance through which the water moves in 1 hour.
3. If $p =$ principal, $r =$ the rate of interest expressed decimally, $n =$ number of years, and $a =$ amount (principal and interest), then \$1 will amount to $(1 + r)$ dollars at the close of 1 year. Show that it will amount to $(1 + r)^2$ dollars at the close of the second year. What will it amount to at the close of the third year? the fourth year? the n th year? What will p dollars amount to at the end of n years?
4. If p dollars when compounded annually for n years amount to $A = p(1 + r)^n$ dollars, show that when compounded semiannually for n years the amount may be found by the formula $A = p\left(1 + \frac{r}{2}\right)^{2n}$.
5. What will the above formula become when the interest is compounded quarterly? when compounded monthly?
6. Compute the compound interest on \$500 at 5 per cent compounded annually for 6 years.
7. Compute the compound interest on \$500 at 5 per cent compounded semiannually for 6 years.
8. What amount of money should a farmer be assured of getting at the end of 40 years for the timber grown on one acre of ground if the land is

worth \$25 per acre when set to timber and if money is worth 4 per cent converted annually, assuming that the land with the timber removed is worth the original cost?

9. It has been found that the amount of food required by an animal for maintenance is not directly proportional to its weight. The following empirical formula gives the relation between the weight of an ox and the amount of food required.

$$E = \frac{W^{0.7}}{10^{0.556}}, \text{ where } E \text{ is the available energy (kt)}$$

of the food required for the maintenance of the animal. W is the weight of the animal. (One kilo-pound unit, (1 kt.) is the amount of heat required to raise the temperature of 7,000 pounds of water 100° .) Show that this formula may be written $\log_{10} E = \log_{10} W^{0.7} - 0.556$.

$$\text{Log}_{10} E = 0.7 \log_{10} W - 0.556.$$

10. Show that the daily food required by a 1,000-pound ox should yield 35 kt. units (kilo-pound units), that is, when $W = 1,000$ and $E = 35$.

11. How many kilo-pound units are required for an ox weighing 2,000 pounds? If the food required by an ox for maintenance varied directly as its weight, what would be the units required for a 2,000-pound ox if a 1,000-pound ox required 35 kt.?

12. How many kilo-pound units are required for an ox weighing 1,250 pounds?

13. If the formula for calculating maintenance rations for sheep is $\log_{10} E = 0.7 \log_{10} W - 0.8$, find the number of kilo-pound units required for the maintenance of a sheep weighing 150 pounds.

14. Assuming that soil grains are spheres and in any particular case have equal diameters, the number of soil grains in a given weight of soil may be found by the formula

$$N = \frac{6W}{\pi d^3 Sg}$$

where W = weight of soil in grams, d = diameter of grains in centimeters, and Sg = specific gravity of soil. Find the number of grains in 1 gram of soil if the diameter $d = 0.164$ centimeter and $Sg = 2.65$.¹

15. Find the soil surface in one gram of the soil given in Problem 14.

NOTE.—Much of the water retained by soils is held there in the form of thin films surrounding the grains.

¹ KING, p. 117.

16. Assuming that a particular soil is formed of equal spherical grains having the closest packing, the spaces left between the grains form capillary tubes whose cross-section is triangular. Show that the area of a minimum cross-section is given by the formula: $A = \left(\sqrt{3} - \frac{\pi}{2}\right)r^2 = 0.1613r^2$. *Hint:* An idea of the capillary tubes formed by soil grains having the closest packing may be obtained by drawing three circles each tangent to the other two. Find the area of the triangle formed by joining the centers of these circles. What part of the three circles is included in this triangle? From this information determine the area of the cross-section of the capillary tubes.

17. If $r = 0.01635$ millimeter find the diameter of the circle equivalent to the capillary tube in Problem 16.

18. In measuring the relative amount of evaporation from plants and a free water surface, a cylindrical porous vessel with a hemispherical cap is used. Derive a formula for finding the convex surface (evaporating surface) of such a vessel in terms of radius and height. (Draw a figure.) What is the convex surface of a cylinder whose height is the same as that of the vessel?

19. Use the formula derived in the above problem in finding the convex surface of the porous vessel when its radius is 12.6 centimeters and its height is 8.37 centimeters. (Let $\pi = 3.14$.)

20. In a bulletin on the study of egg production in the domestic fowl, R. Pearl and F. M. Surface, writing for the U. S. Department of Agriculture, give the following equation:

$$y = 90.47 \left(1 + \frac{x}{9.472}\right)^{19.84} \left(1 - \frac{x}{104.9}\right)^{208.5}.$$

Find the value of y when $x = 10$.

21. Porosity of a soil is defined as the ratio of the open space to the whole space occupied by the soil. If we imagine a soil made up of spherical particles so arranged that the lines joining the centers of eight contiguous spheres form cubes, what is the porosity of such a soil?

22. If, in Problem 16, the spherical particles are so arranged that the lines joining the centers of eight contiguous spheres form oblique parallelopipeds whose edges are equal and whose acute face angles are 60° , what is the porosity of such a soil?¹

23. The porosity of a soil made up of spherical particles packed so that the lines connecting their centers form parallelopipeds whose edges are equal, is given by the formula:

¹ The parts of the eight spheres that form the parallelopiped just complete a single sphere.

$$m = 1 - \frac{\frac{\pi}{6}}{(1 - \cos \theta) \sqrt{1 + 2 \cos \theta}}$$

where m = porosity and θ is the face angle of the parallelopiped. What is the porosity of a soil where $\cos \theta = 0.3420$?¹

24. If the movement of moisture in soil follows the rate law $y^n = kt$, check the following table for alluvial soil, Gila River, where $n = 1.86$.²

$$y^n = kt$$

ALLUVIAL SOIL, GILA RIVER

t = time in minutes	y = height in inches	k
2	1.5	1.05
5	2.4	1.02
10	3.6	1.08
15	4.3	1.01
30	6.3	1.05
60	9.2	1.07

¹ See nineteenth Annual Report of Department of Interior, U. S. Geological Survey, Vol. II, pp. 311 to 312, 1898.

² Cameron, p. 28.

CHAPTER VII

SIMULTANEOUS EQUATIONS

26. Graph of $y = mx$.—The statement y varies as x is expressed by the fundamental relation $y = mx$. For different values of x we may determine corresponding values for y . Let us make a table of values for the function $y = 2x$, giving x different values and determining the corresponding values for y :

$x \dots$	-3	-2	-1	0	1	2	3
$y \dots$	-6	-4	-2	0	2	4	6

To obtain a graphical representation of the function $y = 2x$, plot the pairs of values in this table by using rectangular axes as was done in the second chapter in locating cities. Name the horizontal axis XX' and the vertical axis YY' and use a scale $\frac{1}{2}$ inch = 1 unit. If these points are accurately located they will lie on a straight line.

27. Slope.—The slope of a line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined by $\frac{y_2 - y_1}{x_2 - x_1}$. We shall designate slope by the letter m , then

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (\text{See Fig 4})$$

Expressed in words, the slope of a line joining two points is defined as the difference of the ordinates of the points divided by the difference of their abscissas taken in the same order.

Does this definition of a slope of a line agree with your definition of a slope of a hill? If $x_2 = 4$, $y_2 = 5$, and $x_1 = 2$, $y_1 = 3$, what is the slope of the line?

Let us return to the graph (Art. 26) of the equation $y = 2x$, and

(a) Find the slope of the line segment joining the points $x = 2, y = 4$, and $x = 3, y = 6$. These points may be designated $P_1(2, 4)$ and $P_2(3, 6)$ or simply $(2, 4)$ and $(3, 6)$.

(b) Find the slope of the remaining segments using the other point pairs in the table.

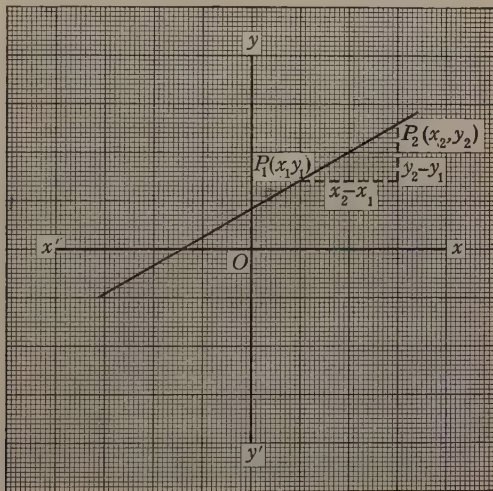


FIG. 4.

Do these line segments lie in a straight line? Give reasons for your answer.

By a method similar to the above, it can be shown that the graph of $y = mx$ or $y = mx + b$ is a straight line.

28. An Equation of a Straight Line Is $y = mx + b$.—On the line AB (Fig. 5), take any point $P(x, y)$ and construct the right triangle HKP . Then the slope of the line AB is

$$\frac{KP}{HK} = \frac{y - b}{x} = m. \quad \text{Why?}$$

From this equation we have:

$$y - b = mx$$

or

$$y = mx + b.$$

Since (x, y) are the coordinates of any point on the line AB , it follows that $y = mx + b$ is the condition that the point $P(x, y)$ lies on this line and $y = mx + b$ is said to be the equation of the line. Notice that the coefficient of x is the slope of the line and b is the intercept on the y -axis.

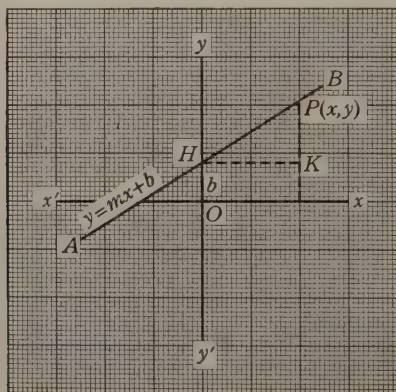


FIG. 5.

Exercise 16

✓ 1. Graph the function $y = 2x + 3$ on the same axes and to the same scale as (Art. 26) $y = 2x$. Make a table of values corresponding to the table (Art. 26) for the function $y = 2x$. Compare the slopes and the Y intercepts of these graphs.

2. If $f(x) = 4x$, find $f(-2)$, $f(-1)$, $f(0)$, $f(2)$, and $f(3)$. With these values for ordinates and the corresponding values of x for abscissas, graph the equation $y = 4x$.

3. On the same axes and to the same scale as Problem 2 graph the equation $y = 4x - 6$. Do these graphs appear to be parallel? What is the value of m for each equation? What are their Y intercepts?

4. See if the point $x = 4$, $y = 10$, lies on the graph of the equation in Problem 3. Does the point $x = \frac{1}{2}$, $y = -4$ lie on this graph? Substitute $(\frac{1}{2}, -4)$ in the equation $y = 4x - 6$. Is the equation satisfied?

5. Given $3x + 2y = 12$. See if the following pairs of values satisfy this equation:

x	-2	0	2	4	6
y	9	6	3	0	-3

Plot these points and draw the graph of the equation. Find other pairs of values which will satisfy the above equation and see if the corresponding points lie on your graph.

6. Given $4x - y = 5$. See if the following pairs of values satisfy this equation:

x	-2	0	2	4	6
y	-13	-5	3	11	19

Continue this table of values by using negative values of x . Plot these points on the same axes and to the same scale as Problem 5.

7. Write the equations in Problems 5 and 6 in the form of $y = mx + b$, that is, solve those equations for y . What are the slopes and the Y intercepts of the graphs of these equations? What are the coordinates of their point of intersection? The *graphical solution* of the simultaneous equations:

$$3x + 2y = 12$$

$$4x - y = 5$$

is the coordinates of the point common to the two lines. Do these coordinates satisfy both of these equations?

8. On the same pair of axes graph $y = x$ and $y = -x$. What is the graphical solution of these equations?

9. Solve graphically

$$\begin{aligned} x - y &= 1 \\ 5x - y &= -7. \end{aligned}$$

10. Find the locus (graph) of $x = 4$. Find the locus of $y = 4$. What is the graphical solution of these lines?

11. Solve graphically

$$\begin{aligned} 2x - 3y &= 1 \\ 5x - y &= 7. \end{aligned}$$

12. Graph the following equations by means of intercepts and find their coordinates of intersection:

$$2x + 3y = 7 \tag{1}$$

$$3x - 2y = 4. \tag{2}$$

Let $x = 0$ in equation (1), then $y = 2\frac{1}{3}$. Locate on the y -axis the point $(0, 2\frac{1}{3})$. Now let $y = 0$, then $x = 3\frac{1}{2}$. Locate on the x -axis the point $(3\frac{1}{2}, 0)$. Draw a straight line through these points. The corresponding points for equation (2) are $(0, -2)$ and $(1\frac{1}{3}, 0)$. Draw a straight line through these points. These two lines cut at the point $x = 2, y = 1$, which is the *graphical solution* of equations (1) and (2). The above method of graphing is known as the *method of intercepts*.

13. Check Problem 12 by solving algebraically the equations

$$2x + 3y = 7 \quad (1)$$

$$3x - 2y = 4. \quad (2)$$

Multiplying equation (1) by 3 and equation (2) by 2 we obtain

$$6x + 9y = 21 \quad (3)$$

$$6x - 4y = 8. \quad (4)$$

Equation (4) subtracted from equation (3) gives

$$13y = 13$$

or

$$y = 1.$$

Substituting $y = 1$ in equation (1) and simplifying, we obtain $x = 2$. The algebraic solution of equations (1) and (2) is, therefore: $(2, 1)$.

Solve Problems 14, 15 and 16 graphically by the method of intercepts and check by solving algebraically:

$$14. 3x - 2y = -4 \quad 15. 3x + 2y = 6$$

$$2x + y = -5 \quad 6x - 2y = 6.$$

$$16. x + y = 1$$

$$5x - 5y = 2.$$

$$17. 9x + y = 5$$

$$3x + 17y = 11.$$

$$18. \frac{x}{6} - \frac{y}{7} = 1$$

$$\frac{x}{3} - \frac{y}{4} = 1.$$

$$19. 3K + 4L = 16$$

$$5L - 2K = 11.$$

$$20. 3x + 2y = 26$$

$$2x + y = 15.$$

$$21. 11P + 16Q = 64$$

$$7P - 12Q = 13.$$

$$22. 7N - 9P = -22$$

$$N = -4.$$

$$23. \frac{y-7}{5} = \frac{x+10}{3}$$

$$\frac{y-4}{2} = \frac{x-8}{3}.$$

$$24. \frac{2x}{5} - \frac{5y}{6} = -\frac{1}{2}$$

$$\frac{x}{6} + \frac{5y}{9} = \frac{5}{8}.$$

$$25. a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2.$$

$$26. x - \frac{(4y-9)}{11} = 5$$

$$\frac{9}{2} - \frac{(x+5)}{3} = -3y.$$

$$27. 4x - 3y = 1$$

$$4y - 3z = -15.$$

$$4z - 3x = 10.$$

$$28. x + y + 1 = 0$$

$$y + z = 4$$

$$z + x = 1.$$

$$29. x + y + z = 6$$

$$2x + 3y - z = 5$$

$$4x + 2y + 2z = 14.$$

$$30. x + y = 10$$

$$x + z = 19$$

$$y + z = 23.$$

$$31. y + z = a$$

$$z + x = b$$

$$x + y = c.$$

32. $2y - 3z = 5$

$5x + y + z = 20$

$x - y = -5.$

33. $2A + B + 2C = 12$

$2A - B + C = -1$

$A + B - C = 2.$

34. In a right triangle the sum of the legs equals 7 and the hypotenuse is 5 units in length. What is the length of each leg?

29. Applications of Simultaneous Equations to Balanced Rations.—As a general proposition, a cow in milk needs about 2 pounds of hay per day per hundredweight or its equivalent, and the balance of her ration should be some grain mixture adjusted to the quantity of and quality of her daily milk yield.

Illustrative Problem

If a 1,000-pound cow under certain conditions needs a ration of 2.21 pounds of protein to 15.9 pounds carbohydrate equivalents, how many pounds of red clover hay and of corn silage should be used in order to produce the above ration?

From the table of Principal Michigan Feeding Stuffs (page 40) we obtain after reducing fats to carbohydrate equivalents, the following amounts of protein, carbohydrates, and fats in 1 pound of the feed under consideration. Notice that the numbers in this table are *not* percentages.

	Protein	Carbo- hydrate	Fats	Carbohydrate equivalent
Red clover hay.....	0.076	0.39	0.018	0.431
Corn silage.....	0.011	0.15	0.007	0.166

Let C = number of pounds of clover hay in ration.

Let S = number of pounds of corn silage in ration.

Then

$$0.076C + 0.011S = 2.21 \quad (1)$$

$$0.431C + 0.166S = 15.9. \quad (2)$$

Solving, we find

$$C = 24.4$$

$$S = 32.5.$$

(In practice, feeds would not be balanced to this degree of accuracy.)¹

¹ The practical dairyman uses the working rule that a dairy cow needs as roughage 1 pound of hay and 3 pounds of silage for each 100 pounds of live weight of the animal. By use of concentrates as oilmeal and grains, he completes the balance.

Exercise 17

1. Solve this illustrative problem graphically by graphing the lines represented by equations (1) and (2) by the intercept method.

2. Balance red clover hay and corn meal for a 1,000-pound cow under the same conditions as in the illustrative problem. Check work graphically.

3. Balance timothy hay and cow-pea meal under same conditions as in Problem 2.

4. Turn to your table of Analysis of Principal Michigan Feeding Stuffs and number the different feeds in their order as, 1, 2, 3, etc. Plot these different feedstuffs with carbohydrate equivalent on the horizontal axis (choose this axis the long way of the paper) and proteins on the vertical axis. Use scale $\frac{1}{2}$ inch = 5 per cent. After locating each feed give it the proper tabular number. Draw a line through the points (0, 0) and (70, 10) where the number 70 is taken along the carbohydrate axis. This line represents a ratio of 7 to 1 of carbohydrate equivalent to proteids. This is approximately the ratio of 15.9 to 2.21. If a mixture of 7 to 1 of carbohydrate equivalents and proteids constitutes a satisfactory ratio for cattle, which of the above feeds satisfy this ratio approximately? With reference to your dietary line what can you determine concerning the position on your chart of the feeds in each of the Problems 1, 2, and 3?

5. Balance timothy hay and wheat bran under the same conditions as in the illustrative problem. Check work graphically.

6. Compound a ration by using one-half of each of the mixtures found in Problems 1 and 2 above. Check your results by adding the amounts of protein in each of the four feeds.

7. (a) Balance corn meal and ground oats.

(b) Compound these feeds with those in the illustrative problem above by taking one-half the amount of each mixture. NOTE.—See if this ration supplies 2.21 pounds of protein and 15.9 pounds of carbohydrate equivalent. Does this compound feed make a better ration than that in the illustrative problem as to the weight of roughage that should be fed per day to a dairy cow?

(c) Compound a ration by taking three-fifths of the amount from the ration in the illustrative problem and two-fifths of the amount from (a).

A TABLE OF PERCENTAGES OF PLANT FOODS IN VARIOUS FERTILIZING MATERIALS¹

The following table includes averages of several analyses

Material	Nitrogen, percentage	Phosphoric acid, percentage	Potash, percentage
Horse manure.....	0.7	0.25	0.55
Sheep manure.....	0.95	0.35	1.00
Hen manure (air dried).....	1.48	1.53	1.46
Seaweed.....	2.33	7.3	2.7
Bat guano—Texas.....	6.47	3.76	1.31
Fish waste.....	9	11	
Dried blood.....	12	1	0.7
Dried meat meal.....	13		
Horn and hoof meal.....	13	5.5	
Garbage tankage.....	3	2	1.1
Cottonseed meal.....	4	2.5	1.7
Linseed meal.....	5.4	1.8	1.2
Nitrate of soda.....	16		
Bone meal.....	4	22.3	
Bone black.....		15	
Potassium nitrate.....	14		45
Kainite.....			12.8
Carnallite.....			9.8
Sylvanite.....			17.4
Sulphate of potash.....			51.8
Muriate of potash.....			50.6
Phosphate rock.....		14	

¹ From "Manures and Fertilizers," by WHEELER.

30. Applications of Simultaneous Equations to Fertilizers.—

The three most important plant foods are nitrogen, phosphoric acid, and potash. These are, therefore, the ingredients commonly considered when purchasing or preparing fertilizers for special purposes. In giving the percentage composition of a particular fertilizer it is customary to give the percentages of these three ingredients in the order named, *e.g.* a 4-8-10 (read four-eight-ten) fertilizer is one containing 4 per cent

nitrogen, 8 per cent phosphoric acid, and 10 per cent potash. A ton of this fertilizer would contain, therefore, 80 pounds of nitrogen, 160 pounds of phosphoric acid, and 200 pounds of potash.

Illustrative Problem

A 7-4-0 commercial fertilizer is made from the following materials:

	Nitrogen, percentage	Phosphoric acid, percentage	Potash, percentage
Horn and hoof meal.....	13	5.5	
Fish waste.....	9	11	

This fertilizer requires 7 per cent of 2,000 pounds or 140 pounds of nitrogen and 4 per cent of 2,000 pounds or 80 pounds of phosphoric acid in 1 ton of the fertilizer. Therefore, if M = number of pounds of horn and hoof meal in 1 ton of this fertilizer and W = number of pounds of fish waste in 1 ton of the fertilizer, then

$$0.13M + 0.09W = 140$$

$$0.055M + 0.11W = 80.$$

Solving these equations simultaneously, we obtain

$$M = 877 \text{ pounds}$$

and

$$W = 288 \text{ pounds.}$$

This fertilizer requires, therefore, 877 pounds of horn and hoof meal and 288 pounds of fish waste in 1 ton. It is evident that a 7-4-0 fertilizer made from these two ingredients would need 835 pounds of *filler*.

NOTE.—*Filler* is any inactive substance such as coal ashes, sand, etc. Since such a filler is a useless expense, its demand may be removed by the use of a low-grade garbage tankage.

Exercise 18

1. Solve the above illustrative problem graphically by the intercept method, that is, let $M = 0$ and find W ; then let $W = 0$ and find M . Use scale $\frac{1}{2}$ inch = 200 units.

2. Make a 7-4-0 commercial fertilizer from the following materials:

	Nitrogen, percentage	Phosphoric acid, percentage	Potash, percentage
Dried blood.....	12	1	0.7
Bone meal.....	4	22.3	

NOTE.—Neglect the 0.7 per cent potash in the dried blood. Will the agriculturist object if the manufacturer donates this potash? What will the manufacturer lose on 1 ton of this fertilizer by such a neglect? Check your results by determining the number of pounds of nitrogen and of phosphoric acid in 1 ton of your mixture.

3. Make a [12-5-0 fertilizer from dried blood and bone meal. Following the same method as in Problem 1 we have

$$0.12B + 0.04M = 240$$

$$0.01B + 0.223M = 100.$$

Solving these equations we obtain

$$\begin{array}{rcl} & \text{Pounds} & \\ \text{Bone meal} & = & 364 \\ \text{Dried blood} & = & 1,879 \\ \text{Total} & = & 2,243. \end{array}$$

Since the total number of pounds exceeds a ton, it is impossible to make a 12-5-0 fertilizer from dried blood and bone meal. It is possible, however, to make a fertilizer from dried blood and bone meal such that the ratio of its nitrogen to its phosphoric acid will be as 12 to 5.

4. (Work Problem 3 before working this problem.)

Given: the ingredients of Problem 3.

Required: Nitrogen: Phosphoric acid = 12:5

Solution:

$$0.12B + 0.04M : 0.01B + 0.223M = 12:5 \quad (1)$$

and

$$B + M = 2,000. \quad (2)$$

From equation (1)

$$60B + 20M = 12B + 268M$$

or

$$48B - 248M = 0$$

and

$$48(2,000 - M) - 248M = 0 \text{ from value of } B \text{ in equation (2)}$$

$$96,000 - 296M = 0$$

or

$$296M = 96,000.$$

$$M = 324.$$

Substituting this value in equation (2) we obtain

$$B = 1,676.$$

Therefore, this fertilizer is composed of

$$\begin{array}{r} 324 \text{ pounds bone meal} \\ \text{and } 1,676 \text{ pounds dried blood} \\ \hline 2,000 \text{ pounds} = \text{total.} \end{array}$$

5. Check Problem 4 by showing that the following statements are true:

	pounds nitrogen		pounds phosphoric acid
324 pounds bone meal contain	12.96	and	72.25
1676 pounds dried blood contain	201.12	and	16.76
	<u>214.08</u>		<u>89.01</u>

$$\text{and that } \frac{\text{nitrogen}}{\text{phosphoric acid}} = \frac{214.08}{89.01} = \frac{12}{5}.$$

In what follows we shall refer to a fertilizer which in general requires a filler (see illustrative problem) as an *a-b-c fertilizer*, while a fertilizer in which nitrogen: phosphoric acid: potash = *a:b:c* (see Problem 4) as a *ratio a-b-c fertilizer*.

6. Mix a 4-7-4 wheat fertilizer from dried blood, bone meal, and muriate of potash (see table of fertilizers).

Ans. in pounds: dried blood = 464.4, bone meal = 606.9, muriate of potash = 151.4, filler = 777.3.

7. Mix a ratio 4-7-4 fertilizer from these same materials and by so doing eliminate the useless filler.

The equations obtained are:

$$\left. \begin{array}{l} \frac{12d + 4b}{d + 22.3b} = \frac{4}{7} \\ \frac{12d + 4b}{0.7d + 50.6m} = \frac{4}{4} \end{array} \right\} \text{ where } \begin{cases} d = \text{amount of dried blood} \\ b = \text{amount of bone meal} \\ m = \text{amount of muriate.} \end{cases}$$

$$d + b + m = 2,000.$$

Ans.: $d = 759$ pounds, $b = 992$ pounds, $m = 249$ pounds.

8. Find how the ratio 4-7-4 fertilizer of Problem 7 compares with the 4-7-4 fertilizer of Problem 6. To do this we may find how many pounds of nitrogen is in 1 ton of the ratio 4-7-4 fertilizer.

$$\begin{aligned} 0.12d + 0.04b &= 0.12 \times 759 + 0.04 \times 992 \\ &= 130.8 \text{ pounds nitrogen.} \end{aligned}$$

Since there are 80 pounds of nitrogen in a 4-7-4 fertilizer, therefore $\frac{80}{130.8} = .61$, and therefore, 0.61 pounds of the ratio 4-7-4 fertilizer is equal to 1 pound of the 4-7-4 fertilizer.

9. Mix a 4-7-4 fertilizer of sheep manure, dried blood, and seaweed. The results of your work will show the following: sheep manure = 3,215 pounds; dried blood = 72 pounds; seaweed = 1,753 pounds.

Why is a 4-7-4 fertilizer from these three materials impossible?

10. Mix a *ratio* 4-7-4 fertilizer from sheep manure, dried blood, and seaweed.

Ans.: Sheep manure = 1,276.8 pounds; dried blood = 29 pounds; seaweed = 695 pounds.

11. See if you can convert the *ratio* 4-7-4 fertilizer of Problem 10 to an equivalent of 4-7-4 fertilizer by comparing the amount of potash in 1 ton of each.

NOTE.—A necessary condition that a *ratio* fertilizer can be mixed is that each material entering into it has one element in it which predominates and which is a different element from the predominant element in the other materials.

12. The determination of the amounts of certain materials in fertilizers may be a simple arithmetic problem. For example, mix a 4-7-4 wheat fertilizer from the following materials:

	Nitrogen, percentage	Phosphoric acid, percentage	Potash, percentage
Nitrate of soda.....	16		
Bone black.....	..	15	
Muriate of potash.....	..	..	50.6

Since this fertilizer must have in it 80 pounds of nitrogen and since the only source of nitrogen is the nitrate of soda, therefore, $\frac{80}{0.16} = 500$ and

we see that 500 pounds of nitrate of soda is required, similarly for other materials.

13. Use the same ingredients as in Problem 10 and make a *ratio* 4-7-4 fertilizer.

Ans. Nitrate = 628.4 pounds, bone black = 1172.9 pounds, muriate = 198.7 pounds.

14. Show that one would use approximately 0.8 as much of the *ratio* 4-7-4 fertilizer of Problem 13 as of the 4-7-4 fertilizer of Problem 12.

15. Mix a 4-7-4 fertilizer of nitrate of soda, bone black, and carnallite. Why is a 4-7-4 fertilizer from these three materials impossible?

16. Mix a *ratio* fertilizer from nitrate of soda, bone black, and carnallite.

Ans. Nitrate = 444.5 pounds, bone black = 829.8 pounds, carnallite = 725.7 pounds.

17. Show that $1\frac{1}{8}$ pounds of this *ratio* 4-7-4 fertilizer is equivalent to 1 pound of the 4-7-4 fertilizer.

18. Mix a *ratio* 1-4-10 fertilizer from bone meal and kainite.

19. What amounts of nitrate of soda and bone meal should be mixed to produce a *ratio* 4-5-0 fertilizer?

20. What amounts of dried blood and sulphate of potash should be mixed to form a *ratio* 5-1-10 fertilizer?

CHAPTER VIII

QUADRATIC FUNCTIONS AND EQUATIONS

31. Methods for Solving Quadratic Equations and Graphing Functions.—Expression of the form $f(x) = ax^2 + bx + c$, where a is not zero, is called a *quadratic function* of x and the equation $ax^2 + bx + c = 0$ is the type form of a quadratic equation. The solutions (roots) of a quadratic equation are easily found if $ax^2 + bx + c$ can be factored by inspection.

Example:

$$x^2 + 11x + 30 = 0,$$

then

$$(x + 5)(x + 6) = 0,$$

and, therefore,

$$x = -5$$

and

$x = -6$ are the roots of
the equations.

Exercise 19

Solve the following quadratic equations by factoring and check your results:

1. $x^2 - 1 = 0$. 2. $x^2 + 9x + 8 = 0$. 3. $y^2 - 20 = y$.

4. $5z^2 + 29z - 6 = 0$. 5. $y^2 = 25$. 6. $x^2 + 25x + 144 = 0$.

When the factors of the quadratic equation are not easily found the roots may be determined by the graphical methods or by completing the square.

Completing the Square.—Notice that in the square $(x + a)^2 = x^2 + 2ax + a^2$ the coefficient of x^2 is unity and the term a^2 may be obtained by taking one-half the coefficient of x and squaring it. Making use of the particular form of this quadratic function, which is a complete square in x , we will make the following quadratic equation a complete square:

Solve $3x^2 - 2x = 5$.

This equation may be written:

$$x^2 - \frac{2}{3}x = \frac{5}{3}.$$

Completing the square in the first member by adding the square of one-half of the coefficient of x which is $(-\frac{1}{3})^2$ and balancing the equation by adding $\frac{1}{9}$ to the right member we have

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{5}{3} + \frac{1}{9},$$

or

$$(x - \frac{1}{3})^2 = \frac{16}{9},$$

and

$$(x - \frac{1}{3}) = \pm \frac{4}{3} \text{ (Assumption XXII)f)}$$

and

$$x = \frac{1}{3} \pm \frac{4}{3}.$$

Simplifying

$$x = \frac{5}{3} \text{ or } -1$$

The student should check these two roots.

Exercise 20

Solve the following equations by completing the square:

1. $5x^2 + 3x - 3 = 0$.

2. $x^2 = 7x$.

3. $x^2 - x - 1 = 0$.

4. $x^2 - \frac{2}{3}x = 32$.

5. $3x^2 + 35 = 22x$.

6. $2(x - 3) = 3(x + 2)(x - 3)$.

7. $8x^2 + x = 30$.

8. $x^2 - \frac{7}{8}x = \frac{1}{2}$.

9. $ax^2 + bx + c = 0$.

10. Given the quadratic function: $T = f(d) = 0.051d^2 + 2.5d$. (This function connects the tons T with the depth d of ensilage in a 16-foot silo.) Complete the following table:

$d \dots\dots$	0	10	15	20	25	30	35	40
$T \dots\dots$								

Plot the points whose coordinates are the above pairs of values. Plot depth on horizontal axis and use scale $\frac{1}{2}$ inch = 5 feet. Plot tons on vertical axis and use $\frac{1}{2}$ inch = 10 tons. Draw a smooth curve through plotted points.

11. Solve the quadratic equation $0.051d^2 + 2.5d = 180$. Explain your answer after examining the curve drawn from the table of values in Problem 10.

12. Plot $T = 0.0405d^2 + 1.82d$. (This function expresses tons in a 14-foot silo.) Use same units as in Problem 10 above.

13. How many tons will a 14-foot silo hold when $d = 40$ feet? By means of the solution of a quadratic equation find the value of d measured from the top down which will give one-half the ensilage in this silo.

14. Check graphically Problem 13 by use of the silo capacity curves in Fig. 6. Use these curves in solving the following problem:

15. A 12-foot silo is filled to a depth of 35 feet. After using 20 feet of silage off the top, how many tons remain? *NOTE.*—Silage, after once settled, stays at the same density after the top has been removed.

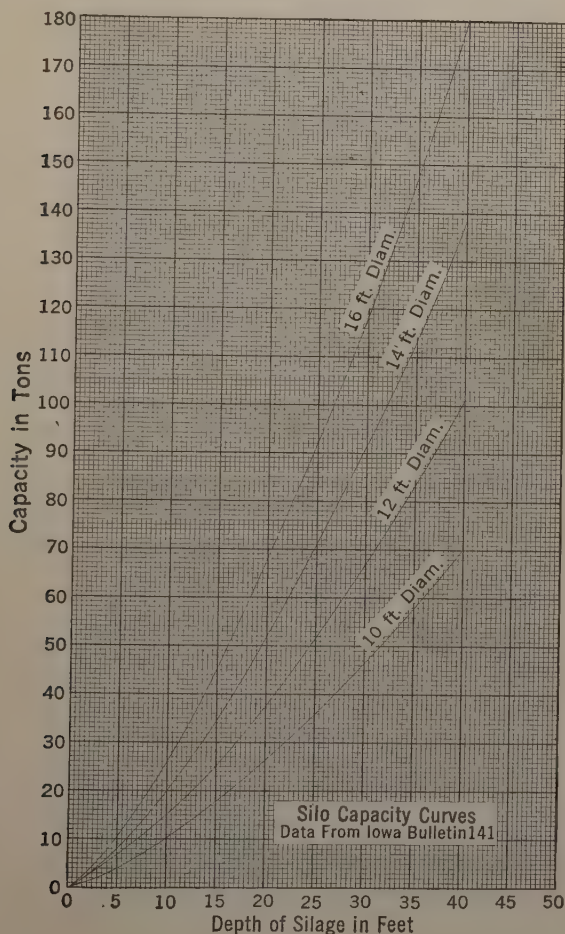


FIG. 6.

Supplementary Problems

1. Assuming that 1 acre of corn will make 15 tons of silage and that one cow eats 35 pounds of silage, per day, how many acres of corn will be necessary to fill a 16-foot silo which is 40 feet deep? How many cows may be fed on this amount for 180 days? Make use of Fig. 6.

2. If, in the above problem, a man starts feeding 20 cows Dec. 1, how deep will the ensilage be on Feb. 1? How many tons will be left?

3. A man harvests yearly 40 acres of corn for silage. How many 16-foot silos will be necessary if none of them can be more than 40 feet high and all are to be built the same height? What will be the height? See Problem 1.

4. A man buys a farm which has three circular silos of the following dimensions, respectively: 16 by 40 feet, 14 by 35 feet, and 10 by 30 feet. How many acres of corn should he plant to fill them if one acre will make 15 tons of silage?

5. By means of your silo capacity curves, check those silo problems which were worked algebraically.

6. (a) If a 16-foot silo is filled to a depth of 40 feet, find the depth measured from the top that gives one-half the total amount of ensilage.

(b) Similarly for a 14-foot silo which is filled to a depth of 35 feet. (c) Similarly for a 12-foot silo which is filled to a depth of 25 feet.

Work and check the following simultaneous quadratic equations. Check 7 and 8 by graphical solution.

$$7. \begin{aligned} T &= 5d^2 + 25d. \\ T &= 100. \end{aligned}$$

$$8. \begin{aligned} T &= \frac{d^2}{100} + d. \\ T &= 60. \end{aligned}$$

$$9. y^2 - 8x = 0.$$

$$y = 4x + 1.$$

$$10. y^2 = 6x.$$

$$y - 3 = -2(x - 1).$$

11. Assuming that soil grains are spherical, of equal size for a particular soil, and have the closest packing, show that the area of a capillary pore varies as the square of the diameter of the soil particles (see Supplementary Problems, Chap. VI, number 16).

12. How is the area of the cross-section of a capillary pore affected if the diameters of the spheres are doubled?

13. The safe load W of a beam supported at the ends and loaded at the center varies jointly as the breadth and the square of the depth and inversely as the length. How is the strength of a beam affected by doubling the breadth? How affected by doubling the depth? The length? Compare the strength of beams 1 and 2; 3 and 4:

$$1. b = 3 \text{ inches, } d = 4 \text{ inches, } l = 15 \text{ feet} \quad 3. 2 \text{ inches} \times 6 \text{ inches} \times L.$$

$$2. b = 4 \text{ inches, } d = 3 \text{ inches, } l = 15 \text{ feet} \quad 4. 3 \text{ inches} \times 4 \text{ inches} \times L.$$

14. The hypotenuse of a right triangle is 100 feet long. Find the other sides, if their ratio is 3 to 4.

15. According to Boyle's law of gases, pressure p times volume v equals a constant k ; that is, $pv = k$. How does the pressure vary with the volume? Show graphically the relation between p and v if $v = 1$ cubic foot when $p = 25$ pounds per square inch.

The equations given in the table express the functional relation of the slope and velocity of farm ditches with particular cross-sections.

Farm ditch	Equation ¹
1	$S = 0.032V + 0.467V^2$
2	$S = 0.022V + 0.356V^2$
3	$S = 0.012V + 0.186V^2$

¹ S = slope (fall) in inches per rod. V = velocity in feet per second.

16. See if the following pairs of values satisfy the equation $S = 0.032V + 0.467V^2$:

$V \dots\dots$	0	0.5	1	1.5	1.7	2
$S \dots\dots$	0	0.133	0.499	1.10	1.40	1.93

Plot these values with velocities on horizontal axis (scale: 2 inches = 1 foot per second) and slope on the vertical axis (scale 2.5 inch = 1 inch). Draw a smooth curve through these points. From your graph read the velocity for a fall of 0.6 inch per rod, for 1.4 inches per rod.

17. Make a table of values V and S for the equation $S = 0.022V + 0.356V^2$ and draw graph. From this graph determine the velocity for a 1 per cent slope. What is V when $S = 1.98$ inch?

The following table gives the equations representing the diameters of tiles (laid with a given slope) necessary to drain a certain number of acres.

Grade, per cent	Equation ¹
0.03	$A = -2.62d + 0.642d^2$
0.05	$A = -3.53d + 0.852d^2$
0.1	$A = -5.82d + 1.24d^2$
1.0	$A = -16d + 3.77d^2$

¹ A = acres; d = diameter of tile in inches.

NOTE.—The above table applies to ground drainage. If surface water also is to be removed (as ponds), the tiles will drain safely only one-half to one-third the number of acres given by the equations. When part of land in the watershed is rolling, count only one-third of such rolling land, in addition to all of the low, flat land in getting the size of tiles to remove ground water only (see Iowa *Bulletin* 78).

18. What size of tile laid to a 0.1 per cent grade will carry the under drainage of 160 acres of flat land?

19. What size of tile laid to a 0.5 per cent grade will carry the under-drainage of 240 acres, two-thirds rolling?

20. What size of tile laid to a 0.1 per cent grade will be required to remove both ground and surface water from a pond whose watershed includes 40 acres?

TABLE OF DRAFT GANG PLOWS MADE IN THE AFTERNOON OF APR. 18, 1906, AT MOLINE, ILLINOIS

Test	Plow	Brand	Average width	Average depth	Total draft, pounds	Draft per square inch, cross-section
1	Oliver gang two-furrow.....	No landslide	28.85	6.77	900	4.61
2	Emerson gang two-furrow...	No landslide	28.05	5.75	675	4.1
3	New Deere gang two-furrow.	N14, no landslide	28.77	6.32	700	3.8
4	New Deere gang two-furrow.	XGP14, with landslide	28.87	6.40	730	3.9
5	New Deere gang two-furrow.	XGP14, no landslide	28.57	5.57	547	3.5
Total for Deere plows.....			86.21	18.29	1,977	11.2
Average for Deere plows.....			28.77	6.10	659	3.7

With reasonable footing a horse can exert a pull equal to about one-tenth of his weight at a speed of $2\frac{1}{2}$ miles per hour, for 10 hours per day and 6 days of the week and keep in condition.¹

21. By use of the above data and the averages in the table, what width furrow $6\frac{1}{2}$ inches deep could be cut by a team weighing 2,800 pounds, using a Deere plow?

¹BAKER, "Roads and Pavements"; also KING, p. 490.

22. If team is capable of plowing a 7 by 12-inch furrow, what load should it be able to haul on a farm wagon on an earth road? (Resistance is 150 pounds per ton.)

23. If a man's diet should consist of 4.5 ounces proteid, 2 ounces fat, and 18 ounces carbohydrate, and be made of whole wheat cereal, milk, and eggs, how many ounces of each are needed? Percentages of ingredients as given by the U. S. Department of Agriculture are: Cereal, 14 per cent proteid, 2 per cent fat, 72 per cent carbohydrate; milk, 3 per cent proteid, 4 per cent fat, 5 per cent carbohydrate; eggs, 15 per cent proteid, 10 per cent fat.

24. How many gallons each of cream containing 30 per cent fat and of milk containing 5 per cent fat shall be mixed so as to produce 10 gallons of cream containing 25 per cent of fat?

25. Can you, by using only two unknown quantities, make a 2-5-10 fertilizer from the ingredients bone meal, muriate of potash, and phosphate rock (14 per cent phosphoric acid)? How many pounds of each of these substances in a ton of this fertilizer? What is the percentage composition of this fertilizer?

26. From the ingredients fish waste, phosphate rock (14 per cent phosphoric acid) and sylvanite, compound a 2-8-8 fertilizer. What is the percentage composition of this fertilizer?

27. It was estimated in 1915 by the chemistry division of the experiment station of Michigan State College, that the fertilizer compounded in Problem 26 would sell at about \$34 per ton. If potash is worth seven-fifths as much as phosphoric acid, and nitrogen is worth four times as much as phosphoric acid, determine the prices per pound that the manufacturer is getting for each of these ingredients.

28. Compound a *ratio* 3-8-5 fertilizer from the ingredients, dried blood, bone, and muriate of potash. (Three ingredients very commonly used in mixing fertilizers.)

29. Solve Problem 26 with potassium nitrate substituted for sylvanite. Explain unusual results.

30. Using bone meal, cottonseed hull ashes (22.5 per cent potash, 8 per cent phosphoric acid) and dried blood, form a 4-8-10 fertilizer. In order to form this fertilizer, how much filler is necessary?

31. The number of tons of ensilage in a silo with a given diameter is a function of the depth of the ensilage. How many tons of ensilage in a 12-foot silo which is filled to a depth of 40 feet if $T = f(d) = 0.0288d^2 + 1.37d$, where T = tons and d = depth? How many tons of silage are fed each month if during the first month 10 feet is fed off the top, the second month 9 feet; and so on?

32. How many tons of ensilage in an 18-foot silo, which is filled to a depth of 35 feet if $T = f(d) = 0.067d^2 + 3.02d$? How many tons remain in the silo after 20 feet have been fed off the top?

33. How many tons of ensilage in a 20-foot silo which is filled to a depth of 45 feet if $T = f(d) = 0.0639d^2 + 4.44d$, where T = tons and d = depth?

34. How many tons of ensilage in a 14-foot silo which is filled to a depth of 28 feet if $T = f(d) = 0.0405d^2 + 1.82d$, where T = tons and d = depth?

The following is a table for the tractive resistance for an ordinary farm wagon on the given surfaces:

Kind of Road Surface	Resistance, Pounds per Ton
Earth road (ordinary condition).....	150
Gravel road.....	50
Macadam.....	25
Sand (ordinary condition).....	200

35. If the friction between a wagon and the roadway varies as the total load on the wheels, and if the friction is 125 pounds when the load is 1,250 pounds, find the friction when the load is 3,125 lbs.

36. The load which a team can haul on a wagon varies inversely as the tractive resistance; if the tractive resistance of an earth road is 150 pounds and that of gravel is 50, what is the relative load for the two roads?

37. If a team can haul $1\frac{1}{2}$ tons over a sand road, what should the team haul (a) on gravel road, (b) on macadam road?

38. What relative loads can be hauled if an earth road is replaced by a macadam?

39. By means of an equation state the following law: "The velocity of flow of water through sands and soils is directly proportional to the effective pressure and inversely proportional to the length of the column through which the flow is taking place" (Darcy's law).¹

¹ KING, p. 262.

CHAPTER IX

PROGRESSIONS

32. Arithmetic Progression.—A succession of quantities so related that each one is obtained by adding a fixed amount to the preceding one is called an *arithmetic progression*. The quantities forming the progression are called its terms. The amount which must be added to any term to get the succeeding one is called the *common difference*.

If a is the first term of an arithmetical progression and d the common difference, then it follows from the definition that

$$a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, a + 6d \quad (1)$$

is an *arithmetical progression*. If the first term $a = 5$, and the common difference $d = 2$, then by substitution, the progression (1) becomes

$$5, 7, 9, 11, 13 \dots$$

33. The n th or Last Term.—In the series $a, a + d, a + 2d, a + 3d \dots$ notice that the coefficient of d is one less than the number of the terms—for example, the coefficient of d in the third term is 2. The n th or general term may be written as $a + (n - 1)d$ and if this is the last term of the series it may be designated as

$$L = a + (n - 1)d \quad (2)$$

In the above progression the first term is 5 and the common difference is 2. The twelfth term is, therefore, $5 + (12 - 1)2 = 27$.

34. Sum of an Arithmetical Progression of n Terms.—The sum of a progression could be found by adding its succes-

sive terms, but this would be a slow process. A formula for such a sum may be derived. It is evident that if L is the last term of the progression, the term just preceding the last is $L - d$ and the term just preceding this is $L - 2d$, and so forth. If, therefore, S_n represents the sum of n terms of an arithmetical progression, then

$$S_n = a + (a + d) + (a + 2d) + \cdots + (L - 2d) + (L - d) + L.$$

Also

$$S_n = L + (L - d) + (L - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

By addition we get

$$2S_n = (a + L) + (a + L) + (a + L) + \cdots + (a + L) + (a + L) + (a + L) = n(a + L).$$

and

$$S_n = \frac{n}{2} (a + L) \quad (3)$$

In the progression 5, 7, 9, 11, . . . we found that the twelfth term was 27. Let us determine the sum of these twelve terms. Substituting in the formula we obtain

$$S_{12} = \frac{12}{2} (5 + 27) = 192.$$

Exercise 21

1. Find the arithmetical progression whose second and fifth terms are 3 and 10, respectively. Then $a + d = 3$, since the second term is 3 and

$$a + 4d = 10; \text{ etc.}$$

2. Sum to ten terms the series $4 + 7 + 10 + 13 \cdots$
3. Find the sum of the consecutive integers from 1 to 1,000 inclusive.
4. Find the fiftieth term of an arithmetical progression whose first three terms are 16, 12, 8.

5. What is the last term of an arithmetical progression of twelve terms if the first term is -2 and the common difference is 2?

6. Find S_n for the series 3, 8, 13 etc. when $n = 20$.

7. A freely falling body falls approximately 16 feet the first second and in each second thereafter 32 feet more than in the preceding second. A

stone dropt from the top of a tower strikes the ground in 4 seconds. How high is the tower and how far did the stone fall the last second?

35. Geometrical Progression.—A succession of quantities so related that the ratio of each one to the preceding one is a fixed number is called a *geometrical progression*. The quantities forming the progression are called its terms. The ratio of any term to the preceding term is called the *common ratio* of the progression and will be represented by r . If a is the first term of a geometrical progression, then it follows from the definition that

$$a, ar, ar^2, ar^3, ar^4, ar^5, ar^6 \dots \quad (1)$$

is a **geometrical progression**. If the first term $a = 2$ and the common ratio $r = 3$, then by substitution the progression (1) becomes

$$2, 6, 18, 54, 162, 486 \dots$$

Notice that the exponents of r in progression (1), namely, 1, 2, 3, 4, 5, 6 are in arithmetical progression. If $a = 2$ and $r = 2$ the progression becomes $2, 2^2, 2^3, 2^4, 2^5 \dots$. This sequence is the one with which we introduced logarithms.

36. The n th or Last Term.—In the sequence $a, ar, ar^2, ar^3, ar^4, ar^5 \dots$ notice that the exponent of r is one less than the number of terms; for example, the exponent of r in the fourth term is 3. The n th or general term may be written as ar^{n-1} and if this is the last term of the sequence it may be designated as

$$L = ar^{n-1} \quad (2)$$

Considering the progression above in which $a = 2$ and $r = 3$, the ninth term, obtained by the formula, is $2 \cdot 3^{9-1} = 2 \cdot 3^8 = 13,122$.

37. Sum of a Geometric Progression of n Terms.—If S_n represents the sum of n terms of a geometric progression, then

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}.$$

Also

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n.$$

By subtraction we get

$$S_n(1 - r) = a - ar^n.$$

Therefore

$$S_n = \frac{a(1 - r^n)}{1 - r}. \quad (3)$$

In the progression 2, 6, 18, 54 . . . we found that the ninth term was 13,122. By means of the equation (3) the sum of these nine terms is

$$S_n = \frac{2(1 - 3^9)}{1 - 3} = 19,682.$$

Exercise 22

- Do the following numbers form a geometrical progression?
 (a) 4, 16, 64, 256.
 (b) 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$.
 (c) 9, -3, +1, $-\frac{1}{3}$, $\frac{1}{9}$.
- Find S_n for the series $2 + 6 + 18$, etc. when $n = 5$.
- Find the sum of the first 10 terms of a geometrical progression whose terms are 16, 8, 4, etc.
- What is the twelfth term of the progression $3 - 6 + 12 - 24$, etc.?
- In shoeing a horse a man received 1 cent for the first nail, 2 cents for the second, 4 cents for the third, etc. What will he receive for 16 nails?
- What will \$200 amount to in 10 years at 4 per cent, compounded annually?
- What will p dollars amount to in n years at r per cent, compounded annually?
- The first term of a geometric progression is 7 and the common ratio is 3. What is the sum of the first 8 terms?
- Find the last term and the sum of the progression 8.1, 2.7, 0.9 . . . to 7 terms.
- Find the sum of the progression: $-\frac{2}{3}$, $\frac{1}{2}$, $-\frac{5}{6}$. . . to 6 terms.
- Given $a = 8$, $r = 2$, $S = 248$, find L and n .
- The fourth term of a geometrical progression is 160 and the ratio is 4. Find the seventh and the fifteenth terms.
- A person has two parents, each of his parents has two parents, and so on. How many ancestors has a person, going back 10 generations, counting his grandparents as the first generation (assuming that each ancestor is an ancestor in only one line of descent)?

CHAPTER X

PERMUTATIONS AND COMBINATIONS

38. Fundamental Theorem.—If a certain act can be performed in m ways and if after this act is completed a second act can be performed in n ways, then the total number of ways in which the two acts can be performed in the order stated is $m \cdot n$.

There are two walks from Agricultural Hall to the library and three walks from the library to the post office. In how many ways may a person pass from Agricultural Hall to the post office by way of the library? If he goes to the library by either of the walks, he has a choice of three walks to complete his journey. Since he has a choice of a second walk from Agricultural Hall to the library and may complete his journey in any one of the three ways, he can go from Agricultural Hall to the post office in $2 \cdot 3$ ways. In general, if one can go from A to B in m ways and from B to C in n ways, then he can go from A to C in $m \cdot n$ ways. With each of the m ways of going from A to B there are n ways of going from B to C . Therefore, the journey from A to C can be made in $m \cdot n$ ways.

Exercise 23

1. Extend the above discussion to include station D with p walks leading to it from station C .

2. Mendel in experimenting with peas chose varieties which differed in respect to (1) the form of the ripe seeds, these being either smooth or wrinkled; (2) the color of the cotyledons, these being yellow or green; (3) the color of the seed coat, this being either white, gray, or brown; (4) the form of the ripe pods, these being either simple inflated, or deeply constricted between the seeds; (5) the color of the unripe pods, these

being yellow or light green; (6) the position of the flowers, either axial or terminal; (7) the length of the stem, either short or tall. In how many ways might nature arrange the first three groups of these differentiating characters, selecting one character from each group? Answer the same question when the seven groups are included.

3. A horse may be driven in a two-horse team hitched abreast in two different ways with reference to position, as right (off) or left (near) side. In how many different ways can six horses be hitched to form teams of two horses each if position is considered?

4. A freshman has a uniform of two pieces, a summer suit of two pieces, a cap and toque, dress shoes and walking shoes. In how many different ways may he dress?

5. In how many ways can a student make the trip home and back if he may go in any one of three ways and return in any one of six ways?

39. Permutations.—Each different arrangement of the individuals in a group is called a permutation, *e.g.*, ab and ba are different permutations of the letters ab ; abc , acb , bca , cab , and cba are the different permutations of the letters a , b , and c .

THEOREM.—The number of permutations of n things taken r at a time is $n(n-1)(n-2)(n-3) \dots (n-r+1)$. This theorem is written symbolically as $Pn, r^1 = n(n-1) \dots (n-r+1)$.

This theorem is made clear by seeing in how many different ways r pockets can be filled with n different objects, assuming that any single object fills a pocket. The first pocket can be filled in n ways and when this is filled the second pocket can be filled in $(n-1)$ ways. By the fundamental theorem above, these two pockets can be filled in $n(n-1)$ ways. When these two pockets are filled, the third pocket can be filled in $(n-2)$ ways and therefore the three pockets can be filled in $n(n-1)(n-2)$ different ways. Continuing this process, the r pockets can be filled in $n(n-1)(n-2) \dots (n-r+1)$ different ways, or the number of permutations of n things taken r at a time is $Pn, r = n(n-1)(n-2)(n-3)(n-4) \dots (n-r+1)$.

¹ Pn, r is also expressed by nPr .

Exercise 24

1. How many numbers of three figures each can be formed from the digits 1, 2, 3, 4, and 5, using each digit but once in a number?

2. In how many ways could an agricultural student choose his instructors for algebra and trigonometry from the 13 instructors in the mathematics department? Assume that he has different instructors in the two courses.

3. Using pockets to illustrate, show that the number of permutations of 5 things taken all at a time is $5 \times 4 \times 3 \times 2 \times 1$ and in a similar manner prove a corollary to the above theorem which is

$Pn, n = n!$ where $n!$ is defined as $n(n-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1$. This symbol $n!$ is called factorial n .

4. Show that the total number of words of 5 letters each formed from the word *ration* cannot exceed 720.

5. How many whole numbers between 10 and 1,000 can be formed from the digits 1, 2, 3, 4, and 5, if no digit is repeated in any one number?

6. How many whole numbers of three unlike digits can be formed from the digits 1, 2, 3, 4, 5, 6?

7. How many signals of four flags each, in vertical order, can be made from twelve flags, all different?

8. In how many ways can wheat, rye, barley, and millet be raised in four fields, each field being used for a single crop?

40. Combination.—A group of things considered without reference to the order of the individuals within the group is called a combination. Thus *ab* and *ba* are different permutations but the same combination.

abc, bcd, cda, dab form all the combinations of four things taken three at a time. Symbolically, this is written $C_{4,3} = 4$. Each of these combinations as *abc* gives rise to six permutations, namely: *abc, acb, bca, bac, cab, cba*, or

$$P_{3,3} = 3 \times 2 \times 1 = 6$$

In general, each of the groups of r things in a combination gives rise to factorial r permutations, and therefore the following theorem is derived.

THEOREM.—The number of combinations of n things taken r at a time is $\frac{1}{r!}$ times the number of permutations of n things taken r at a time, that is,

$$C_{n,r} = \frac{P_{n,r}}{r!} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!}.$$

In deducing the formulas for permutations and combinations we assumed that the n things were different, that is, none were repeated. Formulas that apply when the n things are not all different may be found in more advanced college algebras.

Exercise 25

1. How many combinations can be formed with the letters a, b, c, d , taking three at a time?

2. In how many ways can 5 books be selected from 11 different books?

3. If there are 28 professors in the faculty of a college, in how many ways can the president appoint a committee of three to alter the course of study? In how many of these would A and B be together?

4. From a herd of 15 black and 10 red cows, in how many ways can 5 animals be selected, 2 of which are red and the remainder black?

5. (a) Given that $C_{n,r} = \frac{n(n-1)(n-2) \cdots n-(r-1)}{r!}$,

(b) Show that $C_{n,r} = \frac{n!}{r!(n-r)!}$.

Multiplying both numerator and denominator of the second member of (a) by the consecutive numbers $1, 2, 3, 4, \cdots (n-r)$ inclusive one obtains (b), which gives a second formula for $C_{n,r}$.

6. (a) Given $C_{n,r} = \frac{n!}{r!(n-r)!}$,

(b) Show that $C_{n,r} = C_{n,n-r}$, that is, the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $(n-r)$ at a time.

Hint: Use (a) as a formula and substitute $n-r$ for r .

CHAPTER XI

BINOMIAL THEOREM

41. Binomial Expansion.—By actual multiplication, show that

$$(x + y)^2 = x^2 + 2xy + y^2. \quad (a)$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3. \quad (b)$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \quad (c)$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5. \quad (d)$$

Note that the last expansion may be written as follows:

$$(x + y)^5 = x^5 + 5x^4y + \frac{4 \cdot 5}{1 \cdot 2}x^3y^2 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}x^2y^3 + \frac{2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}xy^4 + y^5$$

Write the first three expansions in a similar form.

Note the number of terms in each of the above expansions, the descending order of the exponents of x , and the ascending order of the exponents of y . Do your observations seem to agree with Newton's Binomial Theorem, which is symbolically expressed as follows:

$$\begin{aligned} (x + y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \dots \\ &\dots + \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \cdot 4 \dots (r-1)}x^{n-(r-1)}y^{r-1} + \dots \\ &\quad + nxy^{n-1} + y^n. \end{aligned}$$

where n is any positive integer and r represents the number of the term.

The expansion of $(x - y)^n$ can be obtained from that of $(x + y)^n$ by putting $-y$ in place of y .

NOTE.—The coefficients in the binominal theorem may be expressed by the formula derived in the study of combinations, *e.g.*, the coefficient of the third term $\frac{n(n-1)}{1 \cdot 2}$ may be written $C_{n,2}$.

Exercise 26

1. Show that $(x + y)^n = x^n + C_{n,1}x^{n-1}y^1 + C_{n,2}x^{n-2}y^2 + C_{n,3}x^{n-3}y^3 + \dots + C_{n,r-1}x^{n-(r-1)}y^{r-1} + \dots + C_{n,n-1}xy^{n-1} + y^n$.
2. (a) By means of Newton's binominal expansion find $(x + y)^2$; (b) use (a) in finding $(x - b)^2$. NOTE.—Substitute $(-b)$ for y .
3. Expand $(\frac{1}{2} + \frac{1}{2})^8$ by the binomial theorem.
4. Expand $(\frac{3}{4} + \frac{1}{4})^4$ by the binomial theorem.
5. Expand $(\frac{3}{4} + \frac{1}{4})^{10}$ by the binomial theorem.
6. Find the sum of the coefficients in the expansion of $(a + b)^n$. Hint: Expand by use of the formula in Problem 1 and then let $a = 1$, and $b = 1$.
7. Find the fifth term in the expansion of $(x + y)^9$ by using the formula for the general term in the expansion of $(x + y)^n$ in the formula above. Check your work by using $C_{n,r-1}$ for determining the coefficient of this term.
8. Repeat Problem 7 in finding the ninth term.

CHAPTER XII

PROBABILITY

42. Definition.—If an event can happen in a ways and fail in b ways, all of which are equally likely to occur, the probability that the event will happen is defined by the fraction $\frac{a}{a+b}$ and the probability that it will fail by $\frac{b}{a+b}$. Referring to all of the equally likely ways in which an event can happen or fail as favorable and unfavorable, the probability that the event will be favorable is the ratio of the favorable cases to the total number of possible cases, favorable and unfavorable.

If p is the probability that an event will happen and q is the probability that an event will fail, then

$$p = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b} \text{ and therefore } p + q = 1.$$

Events are said to be *equally likely* when there is no reason for expecting any one of them rather than any other. When a single coin is tossed, a "head" is equally likely with a "tail," for a head or a tail must turn up and there is no reason for expecting one face to come up rather than the other.

Since there is one way that a coin can fall with a head up and one way that it can fall in which a head fails to be up and both ways are equally likely to occur, the probability that a head will come up when a single coin is tossed is $\frac{1}{2}$.

Although probability as defined above is a definite mathematical quantity it should be thought of as predicting the frequency that an event happens or may be conceived to happen on a very large number of *occasions* or in the *long*

run. A few trials in tossing a single coin will convince the experimenter that in every set of two throws a head will *not* come up once. He will be convinced, however, that as the number of throws increases the ratio of the number of times that a head appears to the total number of throws approaches more and more closely the number $\frac{1}{2}$.

The following exercises require no special knowledge beyond the definition of probability and the fundamental theorem, Chap. X.

Exercise 27

1. A boy draws at random one marble from his pocket that contains seven marbles that are identical with the exception of color, four are black and three are red. What is the probability that he draws a black marble?

The number of favorable selections is four and the total number of possible distinct selections, favorable and unfavorable, is seven. Therefore the probability that the boy draws a black marble is $\frac{4}{7}$.

- (a) If the boy tried this experiment seven times, returning the marble each time and mixing the contents of the pocket thoroughly, would you expect four of the seven trials to be black?
 - (b) If the boy repeated this experiment many times would you expect the ratio of black balls to the total number of trials to approach nearer and nearer to $\frac{4}{7}$?
 - (c) What would you expect with reference to the red marbles?
2. If two identical homogeneous coins are tossed at random, what is the probability:
- (a) That a combination of two heads will come up?
 - (b) That a combination of a head and a tail come up?
 - (c) That a combination of two tails come up?

The favorable and unfavorable ways in which the coins can fall are: (1) head-head, (2) head-tail, (3) tail-head, (4) tail-tail. Any one of these four combinations is equally likely. Since the second and third combinations are alike the probability that the combination of head-tail HT come up is $\frac{2}{4}$ or $\frac{1}{2}$ and that head-head HH come up is $\frac{1}{4}$. The combination of tail-tail TT is identical to that of head-head HH.

3. What is the probability of throwing head *at least* once in two successive throws with a homogeneous coin?

Ans. $p = \frac{3}{4}$.

Where is the fallacy in the following reasoning? The favorable and the unfavorable cases in two throws are HH, HT, and TT while the

favorable cases are HH and HT. Therefore the probability of throwing at least one head in two throws is $\frac{3}{4}$.

The fallacy in this reasoning turns on the question of whether these three cases are equally likely to occur. Examine the analysis of Problem 2 before trying to find the fallacy in the reasoning.

4. If three coins are tossed at random what is the probability:

- (a) That a combination of three heads HHH will come up?
- (b) That a combination of two heads and one tail HHT will come up?
- (c) That a combination of one head and two tails HTT will come up?
- (d) That a combination of three tails TTT will come up?

Are these four cases equally likely to occur?¹

Ans. (a) $\frac{1}{8}$, (b) $\frac{3}{8}$, (c) $\frac{3}{8}$, (d) $\frac{1}{8}$.

Toss three pennies eight times and record what happens each time.

5. If in example four, we express HHH by H^3 , HHT by H^2T , HTT by HT^2 and TTT by T^3 and if we let $(H + T)^3$ be our shorthand expression for a toss of three coins at random, then our analysis of that problem can be expressed symbolically as follows:²

$$\begin{aligned}(H + T)^3 &= HHH + \begin{Bmatrix} HHT \\ HTH \\ THH \end{Bmatrix} + \begin{Bmatrix} HTT + TTT \\ THT \\ TTH \end{Bmatrix} \\ &= H^3 + 3H^2T + 3HT^2 + T^3\end{aligned}$$

This can be interpreted as follows: Three coins are tossed at random (or one coin is tossed three times), with eight *equally possible* arrangements and with two heads and a tail, or two tails and a head appearing three times as often as three heads or three tails.

When three coins are thus tossed what is the probability that all will come up heads; that two will come up heads and one tail; one head and two tails; three tails?

6. Write in detail, as in Problem 5, the analysis of the possible arrangements with reference to head-tail when four coins are tossed and show that your results may be expressed as follows:

$$(H + T)^4 = H^4 + 4H^3T + 6H^2T^2 + 4HT^3 + T^4.$$

Interpret your results in terms of probability.

Ans.: $H^4 = \frac{1}{16}$, $H^3T = \frac{1}{4}$, $H^2T^2 = \frac{3}{8}$, $HT^3 = \frac{1}{4}$, $T^4 = \frac{1}{16}$.

7. With the conventions used in connection with the above problems, the analysis of the possible arrangements with reference to head-tail

¹ The results would be the same if a single coin were tossed three times.

² H and T are not numerical quantities, but simply indicate head and tail.

of n coins tossed at random expressed symbolically is identical to the binomial expansion and therefore the general formula is:

$$(H + T)^n = H^n + C_{n,1}H^{n-1}T + C_{n,2}H^{n-2}T^2 + C_{n,3}H^{n-3}T^3 + \dots \\ \dots + C_{n,r-1}H^{n-(r-1)}T^{r-1} + \dots + C_{n,n-1}HT^{n-1} + T^n.$$

By use of this formula answer the following questions:

- (a) When five coins are tossed at random what is the probability that three of them will come up head and the remaining two come up tail? The answer is obtained immediately by looking for the term that has a T^2 . This is the third term. It must necessarily produce head three times. Now, $C_{n,2} = \frac{5 \cdot 4}{2} = 10$ (since $n = 5$) is the number of favorable cases. $2^5 = 32$, the total number of favorable and unfavorable cases, is easily obtained by the fundamental theorem. The probability therefore, that three of the five coins will come up heads is $\frac{10}{32} = \frac{5}{16}$.
- (b) What is the probability that one only will come up head?

Ans. $\frac{5}{32}$.

- (c) That one only will come up tail?

8. What is the chance of drawing a black bean from an envelope that contains 4 white beans and 8 black beans? Any bean is as likely to be drawn as any other; there are 8 favorable chances and 4 unfavorable chances which are all equally likely; the required probability is therefore $\frac{8}{12} = \frac{2}{3}$.

9. Two beans are to be drawn from an envelope that contains 8 black and 4 white beans. What is the probability that both will be black? Any pair of beans is as likely to be drawn as any other pair. The favorable and unfavorable cases are $C_{12,2}$. (It is less work when the number of combinations is large to use the formula rather than the fundamental theorem.) The number of pairs of black beans that can be selected from 8 is $C_{8,2}$. The probability therefore of drawing 2 black beans is $p = \frac{C_{8,2}}{C_{12,2}} = \frac{14}{33}$.

10. In Problem 9 show that the probability of drawing one white bean and one black bean in a single draw of two beans is $p = \frac{16}{33}$.

Hint: Use fundamental theorem in determining the numerator.

11. A bag contains 7 white marbles, 4 red marbles, and 3 black marbles.

- (a) From this total of 14 marbles how many combinations, taking 6 at a time can be formed?
- (b) From the 7 white marbles how many combinations, taking 2 at a time, can be formed?

- (c) From the 4 red marbles how many combinations, taking 3 at a time, can be formed?
- (d) By means of the fundamental theorem show that 2 white marbles, 3 red marbles, and 1 black marble can be selected from 7 white, 4 red, and 3 black marbles in 252 different ways.
- (e) If 6 marbles are drawn at random, what is the probability that 2 will be white, 3 red, and 1 black?

12. If a flock of chickens consists of 25 black, 30 white, and 50 yellow fowls, show that the probability of a thief getting a chicken of each color in grabbing three in total darkness is $p = \frac{25 \cdot 30 \cdot 50}{C_{105,3}}$.

13. An envelope contains 6 white beans, 4 red beans, and 5 black beans. If 4 beans are drawn at random find the probability:

- (a) That all are white.
- (b) That 2 are white and 2 black.
- (c) That 1 is white, 2 are red, and 1 is black.

14. A hat contains 20 tickets marked 1, 2, 3, . . . 19, 20.

- (a) If numbers 1 and 2 are prize numbers, what is the probability of getting both prizes with a draw of 5 tickets?

The number of combinations of 20 tickets 5 at a time is $C_{20,5}$. The number of ways that three tickets can be selected from 3, 4, 5, 6, . . . 20 is $C_{18,3}$ and with each of these groups of 3 tickets numbers 1 and 2 can be associated to make a group of 5 tickets. Hence the probability that 1 and 2 will be among the 5 drawn is $p = \frac{C_{18,3}}{C_{20,5}}$.

- (b) If 13 and 19 are prize numbers, what is the probability of winning a prize in five draws, that is, what is the probability of drawing either 13 or 19 in five draws?

The probability of *not* drawing either 13 or 19 is $q = \frac{C_{18,5}}{C_{20,5}}$. Therefore, the probability of drawing either 13 or 19 is $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$.

15. A hat contains 25 tickets, 2 of which are blanks and the rest are numbered 1, 2, 3, 4, . . . 23. What is the probability of drawing both blanks on a draw of 3 tickets?

Ans. $p = \frac{23}{C_{25,3}}$.

16. From a pen of 12 leghorn hens 10 of which are laying, one is selected at random, and from a second pen of 16 barred rocks, 12 of which are laying, one is selected at random. What is the probability that both hens are laying hens?

Hint: By the fundamental theorem determine the total number of ways of selecting two hens, one from each pen and also the number of ways of selecting two laying hens, one from each pen.

43. Compound Events.—When two events are such that the happening of one of them at a given trial does not affect the probability of the other, the events are independent, but if the happening of one does affect the probability of the other, the events are dependent.

An envelope contains 3 white and 7 black beans. Suppose we draw 2 beans from the envelope. If we make two drawings, the probability of drawing a white bean on the first trial is $\frac{3}{10}$ and if we get a white bean on this trial then the probability of getting a white bean on the second trial is $\frac{2}{9}$, while if we draw a black bean on the first trial then the probability of getting a white bean on second trial is $\frac{3}{9}$. Therefore these events are dependent. If, however, after the first drawing we replace the bean, then the probability at each trial is the same.

THEOREM.—The probability that two independent events will both occur is the product of the probabilities of the separate events.

For if $p_1 = \frac{f_1}{t_1}$ and $p_2 = \frac{f_2}{t_2}$ are the probabilities of the separate events, t being the total number of events that are equally likely to occur and f the number of favorable cases, then by the fundamental theorem the total number of events that are equally likely to occur in the compound unit is the product $t_1 t_2$ and the number of favorable cases in the compound event is $f_1 f_2$. Therefore the probability of the compound event by definition is $\frac{f_1 f_2}{t_1 t_2}$, which is the product of the probability of the separate events.

In general, if $p_1, p_2, p_3, \dots, p_n$ are the respective probabilities of n independent events, the probability that all will occur is the product of the probabilities of the separate events.

Example: From a pen of 12 leghorn hens, 10 of which are laying, one is selected at random and from a second pen of 16 barred rock hens 12 of which are laying, one is selected at random.

- (a) What is the probability that the white leghorn is a laying hen and the barred rock is not?
- (b) What is the probability that both are laying hens?

Removing a hen from the first pen in no way affects the probability of drawing a hen that is not laying from the second pen. Therefore the separate events are independent. Now the total number of equally likely chances t_1 in the first pen is 12 and the number of favorable cases f_1 , that is, laying hens, is 10. Therefore $p = \frac{10}{12} = \frac{5}{6}$, similarly $p_2 = \frac{12}{16} = \frac{3}{4}$. The probability, therefore, of getting a white leghorn that is laying and a barred rock that is not laying is $\frac{5}{6} \times \frac{1}{4} = \frac{5}{24}$.

Student answer part (b).

Exercise 28

1. By means of the above theorem show that the probability of throwing three heads with three throws of the coin is $\frac{1}{8}$. Compare your work with Problem 5, Exercise 27.

2. Given two bags, one containing 5 white balls and 7 black ones, and the second containing 4 white balls and 11 black ones, what is the probability that in two draws, one from each bag, a white ball will appear each time? Check your result by working this problem by use of the fundamental theorem.

3. Five dice are thrown at random. For convenience we shall name them a, b, c, d, e . Show that the probability that both a and b will fall ace (one spot) is $(\frac{1}{6})^2$; the probability that c, d, e , will not fall ace is $(\frac{5}{6})^3$; and the probability that a, b will fall ace and c, d, e fall something other than ace is $(\frac{1}{6})^2 \cdot (\frac{5}{6})^3$.

4. Referring to Problem 3 and using the notation p = probability that the event happens (a certain face will come up), q = probability that the event fails, and $p + q = 1$, then $p = \frac{1}{6}$, $q = 1 - \frac{1}{6} = \frac{5}{6}$. The final result in Problem 3 may be expressed by $p^2q^3 = (\frac{1}{6})^2(\frac{5}{6})^3$, which is the probability that a certain pair of dice a, b come up ace and the remaining three dice fail to come up ace.

- (a) How many different combinations of two dice each can be formed with five dice?
- (b) Show that the probability that any two dice may come up ace and the remaining three fail to come up ace is $10(\frac{1}{6})^2(\frac{5}{6})^3 = C_{5,2}p^2q^3$.
- (c) Would the final conclusion be different if in place of throwing five dice we had thrown one die five times?

44. Probability That an Event Happens Exactly r Times Out of n Trials.—After working Problems 3 and 4 above, we are prepared to consider the probability that an event will occur a certain number of times in a series of trials, knowing the probability of its occurrence in a single trial. In the above examples we found the probability that an ace will come up exactly twice in a series of five trials, knowing the probability of its occurrence in a single trial to be $\frac{1}{6}$.

THEOREM.—If in a single trial the probability of the happening of an event is p and the probability of its failure is q , then the probability of its happening exactly r times in n trials is $C_{n,r}p^r q^{n-r}$.

By use of the theorem under *compound events*, show that the probability that an event will happen every time during the first r trials is p^r , the probability that the event will fail every time during the remaining $n - r$ times is q^{n-r} ; that it will happen in all of the first r trials and fail in all the remaining $n - r$ trials is $p^r q^{n-r}$.

Instead of choosing the first r trials as the favorable ones, the result would have been the same if any set of r trials had been selected and since there are $C_{n,n-r}$ sets of r trials in n trials, therefore the probability that the event will occur exactly r times in n trials is $C_{n,n-r}p^r q^{n-r}$. **NOTE.**—The sixth problem under the subject of combinations shows that $C_{n,n-r} = C_{n,r}$.

Exercise 29

1. Use the above theorem and determine the respective probabilities when r takes the values 0, 1, 2, 3 . . . $(n - 2)$, $(n - 1)$, n . Add these results in the reverse order in which they were obtained and notice that $C_{n,0} = 1$ since $C_{n,n-r} = C_{n,r}$.

This sum is the binomial expansion

$$(p + q)^n = p^n + C_{n,1}p^{n-1}q^1 + C_{n,2}p^{n-2}q^2 \cdot \cdot \cdot + C_{n,n-1}p^1q^{n-1} + q^n,$$

in which p^n is the probability that the event will be favorable in each of the n trials, p^{n-1} is the probability that the event will be favorable in all but one trial; etc.

2. (a) If a coin is tossed a single time what is the probability that head will come up? That head will *not* come up?
 (b) By use of the above theorem and the information that $p = q = \frac{1}{2}$ in the case of a single toss of a coin, complete the following table which calls for the probability that a certain number of heads will come up in a toss of 10 coins:

Heads.	10	9	8	7	6	5	4	3	2	1	0
Tails.	0	1	2	3	4	5	6	7	8	9	10
Probability.	*	†	‡								

- (c) When the results of (a) and (b) are substituted in the binomial in Problem 1 it becomes

$$\left(\frac{1}{2} + \frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}[1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1].$$

The separate terms within the brackets give the frequency with which 10 heads and no tails, 9 heads and 1 tail, 8 heads and 2 tails, etc., may be expected to come up in the continued tossing of 10 coins at a toss. These results might also be interpreted after introducing 2^{10} in the denominator of each of the terms within the brackets, as follows: $\left(\frac{1}{2}\right)^{10}$ is the probability that 10 heads and no tails may be expected to come up in a single toss of 10 coins, $\left(\frac{1}{2}\right)^{10} \times 10$ is the probability that 9 heads and 1 tail may be expected to come up; etc.

- (d) The probability that 10 heads and 0 tails may be expected to fall in the continued tossing of 10 coins expressed as a percentage is $\left(\frac{1}{2}\right)^{10} = 0.00098 = 0.1$ per cent, approximately. Show that the probabilities for the other combinations of heads-tails expressed as percentages are respectively: 1.0, 4.4, 11.7, 20.5, 24.6, 20.5, 11.7, 4.4, 1.0 and 0.1.
 (e) If 10 coins are tossed 800 times, show that one might expect the above eleven combinations of head-tails to fall respectively as follows: 1, 8, 35, 94, 164, 199, etc., that is, out of 800 throws one might expect all heads to come up once 9 heads and 1 tail to happen 8 times; etc.

$$* C_{10,10} \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{2^{10}}; \quad \dagger C_{10,9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 = \frac{10}{2^{10}}; \quad \ddagger C_{10,8} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 = \frac{45}{2^{10}}.$$

CHAPTER XIII

APPLICATION OF PROBABILITY TO PROBLEMS IN HEREDITY

45. Mendelian Experiment.—Mendel, in his historical experiments with the garden pea, set himself the task of following individual characteristics from one generation to another. He crossed plants having such contrasted characteristics as tall and dwarf stems, wrinkled and smooth seeds, color of ripe seeds as yellow or green, color of seed coat as white or brown, form of ripe pods, simply inflated or deeply constricted between the seeds, difference in the position of the flowers as axial or terminal, and recorded the nature and number of the progeny from the self-fertilized hybrids. He found, for example, when he crossed a tall plant and a dwarf plant that all the progeny (hybrids) were tall. The hybrid plants produced seeds giving plants three-fourths of which were tall like the tall grandparent, and one-fourth dwarf like the dwarf grandparent. When seed from the dwarf plants was planted, it bred true—that is, all the progeny were dwarf. When seed from the tall plants was planted, only one-third bred true, that is, the progeny were tall. The other two-thirds proved to be tall hybrids, for three-fourths of their progeny were tall and one-fourth dwarf. The progeny of the original tall hybrids therefore had the composition of:—one tall: two tall hybrids: one dwarf; that is, they stood in a 1:2:1 ratio.

Definitions of terms needed in the following discussion.

A reproductive cell capable of uniting with another reproductive cell to form a new individual is called a *gamete*. In higher animals and plants these reproductive cells (gametes)

which are capable of union in pairs are of two unlike kinds, namely egg and sperm. With reference to plants we may think of two unlike cells as ovule and pollen grain.

The union of egg and sperm, the fertilized egg, is called a *zygote*, the fertilized egg, and is the beginning of a new life.

It is assumed that the reproductive cells, egg and sperm, carry what are known as *Mendelian characters* or *factors*. These factors are usually designated by letters, as *T* for tall, *C* for color; and so forth. If two cells with opposing characters should unite—for example, the ovule of a tall pea be fertilized by a pollen grain from a dwarf pea—the zygote thus formed would carry the two opposing factors and one by necessity must be obscure. The one which is obscure in such a cross is called a *recessive* and the other is called a *dominant* character. It is the usual convention to designate a dominant character by a capital letter, and the opposing character by the same letter not capitalized. In the above illustration the fertilized ovule would be designated by *Tt*, to indicate that the fertilized ovule carried two factors and that tall dominated dwarf.

Hybrids Possessing One Mendelian Factor.—We shall continue Mendel's analysis of his problem of crossing garden peas. Suppose a tall pea *TT* is crossed by a dwarf plant *tt*, then, according to Mendel's assumption, all the cells of the resulting hybrid plant carry both factors *T* and *t*, and when these hybrid plants are self fertilized, two kinds of gametes are formed both in the ovule (egg) and in the pollen grain (sperm), the one carrying the factor *T*, the other the factor *t*. The problem and solution follow:

- A. Tall hybrids *Tt* are self-fertilized with the result that
T pollen may fertilize *T* ovule and produce *TT*, a tall plant, or
T pollen may fertilize *t* ovule and produce *Tt*, a tall hybrid. Also
t pollen may fertilize *T* ovule and produce *tT*, a tall hybrid, or
t pollen may fertilize *t* ovule and produce *tt*, a dwarf plant.

Before going further, review Problem 2, Art. 42, which has to do with the two factors, head-tail, when two coins are

tossed at random. Assuming that the factors T and t operate according to the theory of probability (chance):

- (a) Is any one of the four possibilities equally likely to occur?
- (b) If we assume that the result is the same whether T pollen fertilizes t ovule or t pollen fertilizes T ovule, what is the probability that the progeny from these hybrid plants will be tall hybrids Tt ?
- (c) That they will be pure tall plants TT ?
- (d) That they will be dwarf plants tt ?

The above problem and its solution stated symbolically is Tt crossed by $Tt = Tt \times Tt = TT + 2Tt + tt$. Expressed in words, the number of pure tall plants equals the number of dwarf plants and the hybrids equal twice the number in either of the other groups. The dwarf plants can be segregated. The pure tall TT and the hybrid plants Tt are alike in appearance, but when bred again produce both tall and dwarf progeny. Note that when the dwarf plants have been segregated the individuals that are left have the genetic make-up ($TT + 2Tt$).

B. Algebraic Solution of Problem A.—Since the ovule and the pollen carry both factors T and t , in place of the notation Tt use $T + t$. Then $Tt \times Tt$ becomes

$$(T + t)(T + t) = TT + 2Tt + tt,$$

which is the conclusion reached by the first method.

The recombinations of characters according to a definite law as shown by the crossing of peas is known as Mendel's law. In seed plants the important factor that determines the character of the progeny is the kind of pollination that takes place.

Exercise 30

1. In a so called open pollinated crop like corn, individuals with genetic make up ($YY + 2Yw$) are mated at random. YY represents

pure yellow and Yw represents a hybrid between yellow- and white-colored corn. Determine the classes of grains and their ratios that will appear in the offspring by the mating of $(YY + 2Yw) \times (YY + 2Yw)$.

Solution.— $YY + 2Yw$ becomes by the algebraic method $(Y + Y) + 2(Y + w)$. Therefore the cross becomes

$$\begin{aligned} [(Y + Y) + 2(Y + w)]^2 &= (Y + Y)^2 + 4(Y + Y)(Y + w) + 4(Y + w)^2 \\ &= 4YY + 4(2YY + 2Yw) + 4(YY + 2Yw + ww) \\ &= 16YY + 16Yw + 4ww \\ &= 4(4YY + 4Yw + ww). \end{aligned}$$

Therefore the types of progeny and their ratios are:

$$YY:Yw:ww = 4:4:1.$$

2. The illustrative problem with the peas shows that $(Tt) \times (Tt)$ produces types of progeny and ratios as follows: $TT:Tt:t = 1:2:1$. Similarly with corn $(Yw) \times (Yw)$ produces $YY:Yw:ww = 1:2:1$. When the pure white has been cast out and the hybrids $(YY + 2Yw)$ mated they produced the following types of progeny and ratios: $YY:Yw:ww = 4:4:1$. Notice that in this generation (third) we have the same types as in the preceding generation but in a different ratio. By continuing the process of excluding the pure white corn, show that the following table is correct for the fourth, fifth, and sixth generations.

Generation	2	3	4	5	6		100		n
YY	1	4	9	16	25		9,801		$(n - 1)^2$
Yw	2	4	6	8	10		198		$2(n - 1)$
ww	1	1	1	1	1		1		1
Total.....	4	9	16	25	36		10,000		n^2
Percentage.....									

- In each generation the hybrids Yw are what per cent of the type YY ?
- By this method of breeding will the types Yw and ww ever disappear absolutely?
- In the hundredth generation what is the chance of pure white corn appearing? In rapidly developing classes the hundredth generation might be reached quite soon.

3. If the population of corn of any generation n is $aYY + bYw + cww$, where a, b, c are integral numbers or zero, show that the next

generation $(n + 1)$ is $(b + 2a)^2YY + 2(b + 2a)(b + 2c)Yw + (b + 2c)^2ww$ if open pollination takes place.

Hint:

$$[a(Y + Y) + b(Y + w) + c(w + w)]^2 = \text{what?}$$

4. (a) In the formula given in Problem 3 let $c = 0$, $b = 2$, and $a = 1$, and by so doing check Problem 1.
- (b) By giving a , b , and c values such that the population of the third generation, as given in the table in Problem 2, is obtained, find the population in the fourth generation.
5. If in the formula in Problem 3, $a = c$ and $b =$ any positive integral number or zero, show that the formula will give: $YY:Yw:ww = 1:2:1$. Does this result seem reasonable to you?

Hybrids Possessing Two Mendelian Factors.—Experiments with guinea pigs show that a rough coat is dominant to smooth, a colored coat to albino conditions. When a rough white guinea pig ($RRcc$) is crossed with a smooth black one ($rrCC$), the hybrids are rough black ($RrCc$). R represents the factor for roughness, C represents factor for color. Now these hybrids produce, in the egg and in the sperm, cells (gametes) which represent all possible combinations that can be formed from these factors taking one factor from each pair, *viz.*, RC , Rc , rC , rc . When the sperm fertilizes the egg sixteen equally likely possibilities arise according to the fundamental theorem. These are easily written out as follows:

1. RC sperm and RC egg gives $RRCC$, a rough colored animal.
2. RC sperm and Rc egg gives $RRCc$, a rough colored animal.
3. RC sperm and rC egg gives $RrCC$, a rough colored animal.
4. RC sperm and rc egg gives $RrCc$, a rough colored animal.
5. Rc sperm and RC egg gives $RRCc$, a rough colored animal.
6. Rc sperm and Rc egg gives $RRcc$, a rough albino animal.
7. Rc sperm and rC egg gives $RrCc$, a rough colored animal.
8. Rc sperm and rc egg gives $Rrcc$, a rough albino animal.
9. rC sperm and RC egg gives $RrCC$, a rough colored animal.
10. rC sperm and Rc egg gives $RrCc$, a rough colored animal.
11. rC sperm and rC egg gives $rrCC$, a smooth colored animal.
12. rC sperm and rc egg gives $rrCc$, a smooth colored animal.
13. rc sperm and RC egg gives $RrCc$, a rough colored animal.

14. rc sperm and Rc egg gives $Rrcc$, a rough albino animal.
15. rc sperm and rC egg gives $rrCc$, a smooth colored animal.
16. rc sperm and rc egg gives $rrcc$, a smooth albino animal.

An analysis of results will show:

- 9 rough black animals
- 3 rough white animals
- 3 smooth black animals
- 1 smooth white animal.

These results may be obtained by what is known as the *checkerboard scheme*.¹

SPERM				
	RC	Rc	rC	rc
RC	$RRCC$	$RRCc$	$RrCC$	$RrCc$
Rc	$RRCc$	$RRcc$	$RrCc$	$Rrcc$
rC	$RrCC$	$RrCc$	$rrCC$	$rrCc$
rc	$RrCc$	$Rrcc$	$rrCc$	$rrcc$

Algebraic Method.—The results obtained by the checkerboard method of a cross of hybrids $RrCc$ can be obtained by means of algebra as follows: $(R + r)(C + c) = RC + Rc + rC + rc$. Give all the possible combinations that can be formed from the factors $RrCc$ by taking one factor from each pair. Since any one of these four combinations of the sperm can unite with any one of the four combinations of the egg, there are sixteen equally likely possibilities which can be found algebraically by the cross

$$(RC + Rc + rC + rc) \times (RC + Rc + rC + rc).$$

This polynomial is the first column as well as the first row of the checkerboard. If one knows the rule for squaring a polynomial the above cross can be obtained by inspection.

¹ See BABCOCK and CLAUSEN, "Genetics in Relation to Agriculture," 1st Edition, p. 87.

Hybrids Possessing Three Mendelian Factors.—For hybrids possessing three Mendelian factors the 8 combinations of the letters $RrCcSs$ (Ss for hair short-long) taking one factor from each pair are obtained from the product $(R + r) (C + c) (S + s)$ and the square of this product gives the 64 equally likely possibilities of a cross of $RrCcSs$

CHAPTER XIV

STATISTICS

46. Definitions and Formulas.—Since there are many kinds of averages such as simple arithmetic mean, geometric mean, median and mode, it is necessary to define the kind of average we have in mind before we make use of the term in any discussion. Let us suppose that in a certain small town *A*, all the inhabitants with but one exception are *factory hands* who have nothing put aside for a rainy day. One individual in this town, however, is a multimillionaire. If the wealth of the town were divided equally among its citizens, each would have thousands of dollars. Is the average wealth of citizens thousands of dollars?

Arithmetic Average.—The arithmetic average or arithmetic mean of several measurements is defined as the sum of the several measurements divided by the number of these measurements.

If $X_1, X_2, X_3, X_4 \dots X_n$ represent N measurements and a their arithmetic mean, then this definition is expressed by the formula $a = \frac{X_1 + X_2 + X_3 + \dots + X_n}{N} = \frac{\Sigma X}{N}$. The symbol ΣX means the sum of all such quantities as X .

Illustrative Problems

1. (a) In the data given in Chap. II, under the heading "Holstein-Friesian Advanced Registry," the pounds of milk produced in four successive milkings are 27.4, 25.2, 25.2, and 25.8. Show that the arithmetic mean of the four milkings is 25.9.
- (b) At the close of the data given in the Holstein-Friesian problem the experimenter makes the following statement: "Average

per cent fat for seven days is 3.694 pounds." Is this an arithmetic average?

2. The lengths of 100 ears of corn are given in the following table. These measurements were made in centimeters:

Length.....	8	9	10	11	12	13	14	15	16	17
Number.....	4	4	9	16	7	25	20	8	6	1

This table was made by placing all the ears of the same length¹ in the same pocket. There were 4 ears 8 centimeters in length, 4 ears 9 centimeters in length, 9 ears 10 centimeters in length; etc. The arithmetic mean by the above definition is found by adding the lengths of the 100 ears and dividing the sum by 100. It is evident, however, that the same result can be found as follows:

Measurements, X	Frequency, f	Product, fX
8	4	$4 \times 8 = 32$
9	4	$4 \times 9 = 36$
10	9	$9 \times 10 = 90$
11	16	$16 \times 11 = 176$
12	7	$7 \times 12 = 84$
13	25	$25 \times 13 = 325$
14	20	$20 \times 14 = 280$
15	8	$8 \times 15 = 120$
16	6	$6 \times 16 = 96$
17	1	$1 \times 17 = 17$
	$\Sigma f = 100$	$\Sigma fX = 1,256$

The arithmetic mean $a = \frac{\Sigma fX}{\Sigma f} = \frac{1,256}{100} = 12.56$. The length 8 is said to have a weight of 4 and the length 13 a weight of 25. An arithmetic mean obtained from weighted items is called the *weighted arithmetic average*. The student should compare the definition of an arithmetic mean as given by the formula $a = \frac{\Sigma X}{N}$ with the one given by formula $a = \frac{\Sigma fX}{\Sigma f}$.

¹See class intervals on p. 115.

A slightly different use of the word *weight* is made when an instructor *averages* all of the *marks* of the student for the term, except the grade of the final examination and gives this average a weight, let us say, of four and the final grade a weight of one. The sum of the two weighted items divided by five, in this case, gives the weighted arithmetic average.

Exercise 31

1. If an instructor gives a weight of three to the student's term marks with the exception of his final, and this is given a weight of two, and if the student has a term average of 90 per cent and a final grade of 70 per cent, what grade will the instructor report to the registrar?

2. Taken from a U. S. Civil Service Examination:

Subjects	Weights
1. Physics, technics, and mechanical drawings.....	60
2. Optional subject.....	40
Total.....	100

If 10 is the unit of measure, what is the weight of item 1? Of item 2?

3. Taken from the U. S. Civil Service Examination:

Junior Crop and Live Stock Estimator. To perform miscellaneous statistical work in the office of the Federal Agricultural Statistician of the state to which assigned. Subjects and weights: Competitors will be rated on the following subjects, which will have the relative weights indicated:

Subjects	Weights
1. Statistical calculations and methods.....	40
2. General agriculture.....	60
Total.....	100

A competitor wrote a perfect paper on part 1 of the examination and answered five out of eight of the equally weighted questions in part 2. What was his grade for parts 1 and 2 combined?

4. Show that the arithmetic mean of 28.3, 25.4, 37.1, 26.5, 29.3, and 35.2 is 30.3. Represent these numbers by points on a horizontal line in the manner that one locates abscissas on the x -axis. A certain number of these points will lie to the left of the arithmetic mean and the remainder to the right. If 30.3 is taken as the origin, then deviations¹ to the left will be considered as negative and deviations to the right as positive. Show that the deviations of these six numbers from the origin are respec-

¹ Given the arithmetic mean a of several numbers $X_1, X_2, X_3, \dots$ the deviation of each number from the mean is $X_1 - a, X_2 - a, X_3 - a, \dots$

tively: -2.0 , -4.9 , 6.8 , -3.8 , -1.0 , and 4.9 , and that their algebraic sum is zero.

5. In Problem 4 you have seen that the algebraic sum of the deviations of the six numbers from their arithmetic mean is zero. Show that the algebraic sum of the deviations of N numbers from their algebraic mean is zero.

Hint: Start with the definition $\frac{X_1 + X_2 + X_3 + \cdots + X_n}{N} = a$.

Median.—The heights and weights of the first eleven freshmen that appeared on a certain date for physical examination follow:

Heights, inches.....	66	71 $\frac{1}{4}$	69	66 $\frac{1}{4}$	65 $\frac{1}{4}$	71 $\frac{3}{4}$	67 $\frac{1}{2}$	67 $\frac{3}{4}$	73 $\frac{3}{4}$	74	72 $\frac{1}{4}$
Weights, pounds.....	133	124	126	130	131	165	147	120	156	159	153

Height and weight are said to be characteristics of the individual. Arranging these eleven students according to the characteristic, weight, and numbering them in this order we have:

Ordered arrangement.....	1	2	3	4	5	6	7	8	9	10	11
Weight.....	120	124	126	130	131	133	147	153	156	159	165

The student occupying the median position as regards weight is in the sixth position in this ordered arrangement. There are just as many students below him as above in the arrangement.

The median of a set of values of a variable is that value that occupies the middle position when these values are arranged according to magnitude. If the set of values is composed of an even number of individual things, then the middle position *is occupied by no one individual thing* and the median may be taken as the arithmetic mean of the two measurements standing nearest the middle.

Exercise 32

Arrange the eleven students in the above table according to the characteristic height and determine whether the student of median height is the

student of median weight in this group of eleven students. Do your observations indicate anything with reference to the height of an individual being a function of his weight? Refer to the correlation table of heights and weights that follows before venturing an answer.

Mode.—In *Bulletin* 119 of the University of Illinois Agricultural Experiment Station by Davenport and Rietz, the following frequency table of the lengths of ears of corn is given:

Length of ears in inches.....	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
Number of ears or frequency.....	1	0	1	0	2	3	9	8	12	19

Length of ears in inches...	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0
Number of ears or frequency.....	32	40	67	63	38	21	8	2	1

The authors state that these 327 ears constitute a *random sample* of a crop of corn that was planted from seed ears each being 10 inches in length. They speak of the modal length of these ears as that length that appears most frequently which in this sample is 9.0 inches. *La mode* means *the style*. Now, if a square-toed shoe is the style, one may expect to meet that cut of shoe on the street more often than the pointed toe. So also, since there are more ears in the above random sample that are 9 inches long than of any other length, one may expect to meet this modal ear more often than any other when the ears are selected at random.

Exercise 33

1. Find the arithmetic mean length of the 327 ears of corn in the above frequency table.
2. What is the modal length of ear in the selection of the 100 ears given in the table in Illustrative Problem 2, Art. 46?
3. After studying the correlation table that follows, give the modal weight and the modal height of the 365 students therein considered.

47. Correlation.—The data included in the following table were taken from the records in Michigan State College that covered the physical examinations of freshmen men for the year 1922-1923. From the entering class the height and weight of the first 365 men to be examined were tabulated as follows: A form with heights along the top and weights along the left margin was made on cross-section paper, the first student, whose height was 66 inches and weight 133 pounds was recorded by a score (a short line) in the square under 66 and opposite 133. Similarly, scores were made for the men whose examination followed. This table was then converted into the

A CORRELATION TABLE OF HEIGHTS AND WEIGHTS OF 365 FRESHMEN
(Height, inches (X))

$Y \backslash X$	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	Total
	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	Total
198	..	..	..	..	..	..	..	..	..	2	..	..	1	..	1	..	..	..	..	4
193	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	1
188	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1	..	..	..	..	2
183	..	..	..	..	..	..	..	..	1	..	2	..	1	..	..	..	..	..	..	4
178	..	..	..	..	..	..	1	1	..	2	1	..	..	..	..	..	..	..	..	5
173	..	..	..	..	..	..	..	..	2	1	1	..	..	..	..	..	..	..	..	4
168	..	..	..	..	1	..	1	1	1	1	2	..	1	1	..	..	..	..	..	9
163	..	..	..	..	..	1	2	3	..	2	5	3	1	1	..	..	..	..	..	18
158	..	..	..	1	..	3	1	1	8	3	3	1	2	..	..	..	..	..	..	23
153	..	..	..	..	2	2	4	4	9	4	7	3	3	..	..	..	..	1	..	39
148	..	..	..	..	2	7	8	11	4	12	4	1	..	..	..	..	..	..	..	49
143	..	..	..	1	4	5	10	7	5	8	3	1	1	..	..	..	..	..	..	45
138	1	..	..	1	6	3	9	7	7	6	1	..	..	..	..	..	..	1	..	42
133	..	1	..	4	4	5	7	7	12	2	4	1	..	..	..	..	..	..	1	48
128	1	..	..	2	5	4	5	4	7	2	1	..	1	..	..	..	..	..	..	32
123	1	..	..	2	6	4	5	3	2	1	1	..	..	..	..	..	..	..	..	25
118	..	..	3	1	1	1	1	2	..	..	..	..	..	..	..	..	..	..	..	9
113	..	..	1	..	..	1	1	..	1	..	..	..	..	..	..	..	..	..	..	4
108	..	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	2
Total	3	1	5	10	20	30	43	53	65	36	50	25	11	7	3	0	0	2	1	365

one below by replacing the total score in each square by the corresponding number. Each square gives immediately the number of students of a certain height and weight. Each number at the bottom and each number at the right of that table gives respectively the total number of men of a given height and the total number of men of a given weight. The table shows, for example, that 65 men had a height of 69 inches and 49 men had a weight of 148 pounds. Such an arrangement of data is called a *correlation table*.

Class Interval.—By examining the correlation table one sees that these students were placed in classes differing in height by 1 inch and differing in weight by 5 pounds. Of course, these class intervals need not be chosen thus. In deciding on an interval one examines the range of values under consideration. In the case of weights, the range is 90 (the difference between the weights at the head and at the foot of the column of weights). An interval of 5 pounds gives 19 classes. If a smaller number of classes seems desirable, a larger class interval is selected. Each weight in the column of weights represents the middle weight of that class. For example the class represented by 148 pounds includes all students whose weights fall within the limits 145.5 and 150.5. Similarly, the class represented by 153 pounds includes all students whose weights fall within the limits 150.5 and 155.5; etc. If the weights of an even number of students should fall on one of the limits as 150.5, one half should be placed in the class represented by 148 and the other half in the class represented by 153. Should the number be odd, then we shall agree to place the odd one in the upper class. For example, suppose five students should have a weight of 150.5 pounds, then two would be placed in the class represented by 148 and three in the class represented by 153.

Exercise 34

1. Turn to the correlation table of height and weight of students and determine whether the median and the mode fall in the same class interval

48. Computation of Arithmetic Mean by Use of Guessed Average.—After examining the table of values—for example, the pounds of milk in the above correlation table—we guess the arithmetic average of these numbers. If the algebraic sum of the deviations of the numbers from the guessed average is zero, we know by Problem 5, Exercise 31, that our guess is the correct mean. If it is not zero, then we must correct our guess by the arithmetic average of the deviations.

Illustrative Problem

Compute the arithmetic mean of the numbers that represent pounds of milk in the above correlation table by making use of the guessed average of 25.0.

Class, X	Deviation from guessed average, d_g	Frequency, f	Deviation $\times$ frequency, fd_g
23.4	-1.6	1	-1.6
23.8	-1.2	2	-2.4
24.2	-0.8	2	-1.6
24.6	-0.4	1	-0.4
25.0	0.0	3	0.0
25.4	0.4	7	2.8
25.8	0.8	7	5.6
26.2	1.2	2	2.4
26.6	1.6	2	3.2
27.0	2.0	0	0.0
27.4	2.4	1	2.4
		$\Sigma f = 28$	$\Sigma fd_g = 10.4$

Since the algebraic sum of the deviations is not zero, we must find the arithmetic mean deviation which is $\frac{10.4}{28} = 0.37$ and add this to the guess. Our true mean is $25.0 + 0.37 = 25.37$. The student can satisfy himself as to the correctness of the above method by representing the measurements by points on a line and considering first 25.37 as origin when the

algebraic sum of the deviations of the measurements from this mean is zero; second, with 25.0 as origin, the algebraic sum of the deviations from it is 10.4. He will see that the guessed mean missed on an average by $\frac{10.4}{28} = 0.37$ of being the correct mean value and because of this fact it must receive a correction of this amount.

The above explanation when stated in a general way is: If $X_1, X_2, X_3 \dots X_n$ are measurements with frequencies $f_1, f_2, f_3 \dots f_n$, if a and G are their arithmetic mean and guessed mean and if $d_1, d_2, d_3 \dots d_n$ the deviations of these measurements from G , then $a = G + \frac{\Sigma fd}{\Sigma f}$; that is, the arithmetic mean value is equal to the algebraic sum of the guessed mean value and the arithmetic mean value of the deviations from this guessed mean.

The proof of this theorem follows:

$$X_1 - G = d_1 \text{ and } f_1 X_1 = f_1 G + f_1 d_1.$$

Similarly,

$$\begin{aligned} f_2 X_2 &= f_2 G + f_2 d_2. \\ f_3 X_3 &= f_3 G + f_3 d_3 \dots + f_n X_n = f_n G + f_n d_n. \end{aligned}$$

If the sum of the first members of these equations is equated to the sum of the second members, then $f_1 X_1 + f_2 X_2 + f_3 X_3 \dots + f_n X_n = G(f_1 + f_2 + f_3 + \dots + f_n) + f_1 d_1 + f_2 d_2 + f_3 d_3 + \dots + f_n d_n$. Dividing each member of the equation by the total number of measurements Σf we have

$$\begin{aligned} \frac{f_1 X_1 + f_2 X_2 + f_3 X_3 + \dots + f_n X_n}{\Sigma f} &= \frac{G(f_1 + f_2 + f_3 \dots + f_n)}{\Sigma f} \\ &+ \frac{(f_1 d_1 + f_2 d_2 + f_3 d_3 + \dots + f_n d_n)}{\Sigma f} \end{aligned}$$

or $a = G + \frac{\Sigma fd}{\Sigma f}$, since first member is the definition of an arithmetic mean value, a , and the second term in the last

member of the equation is the arithmetic mean value of the deviations of the measurements from the guessed average.

Exercise 35

1. (a) Turn to the Holstein-Friesian record in Chap. II and make yourself a frequency table of *butter-fat percentages*. Provide for 22 classes, beginning with 5.1 per cent. Let the limits of this class be 5.15 and 5.05—each interval (class) having a range of 0.1 per cent. When you have finished your table compare it with the following:

FREQUENCY TABLE

Class.....	5.1	5.0	4.9	4.8	4.7	4.6	4.5	4.4	4.3	4.2	4.1
Frequency.....	1	0	0	0	0	0	1	1	1	2	0

Class.....	4.0	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	3.0
Frequency.....	0	2	2	4	1	1	4	5	2	0	1

- (b) By means of this frequency table compute the arithmetic mean per cent butter fat. Assume that the arithmetic mean is 3.4 per cent and put your work in the form used in computing the arithmetic mean of pounds of milk.

2. A commission merchant sells oranges in lots to the highest bidder. He must telegraph returns immediately at the close of the sale on the average price per box. The clerk uses the guessed average method for finding the arithmetic mean. Before the sale begins he guesses that the oranges will average \$5 per box. The sale is as follows:

18 boxes at \$4.90

25 boxes at 5.15

8 boxes at 5.10

13 boxes at \$4.70

7 boxes at 5.10

8 boxes at 5.16

What was his report?

3. The following table of data is a *random sample* of a crop of corn. The lengths are in centimeters, each measurement representing an ear. Make a frequency table of lengths and by the guess method determine the arithmetic mean of these lengths.

LENGTH OF EARS, CENTIMETERS

13	14	12	13
14	9	18	8
16	10	10	13
14	14	14	14
12	12	12	12
17	12	12	12
16	12	12	11
11	11	11	11
12	12	12	15
11	11	11	13
18	12	13	13
11	11	11	14
14	11	11	14
11	11	13	13
13	11	11	13
12	13	15	10
16	14	13	17
13	13	13	16
9	17	9	8
17	17	17	7
18	13	13	8
10	11	13	15
10	16	10	10
14	12	12	9
14	12	12	9

What is the length of the modal ear? The arithmetic mean gives a good average value in general of the variety or race. Which one of the different averages gives the prevailing type of the variety? Is the general farmer interested most in the arithmetic mean or the mode?

49. Deviations from an Average.—Returning to the frequency table on page 113, we find that the seed ears of that population of corn were 10 inches in length. We shall speak of the parent corn as of ideal type. We found that the arithmetic mean of the 327 ears was 8.83 inches in length. Then there was the ear 9 inches in length that appeared most frequently (modal ear) which is the prevailing type. In

order to speak intelligently about any character of a population, whether that population be ears of corn, people, or what not, one must know how the individuals deviate from the average. If all that we know about the inhabitants of a small town is that the arithmetic mean wealth of the population is \$100,000 we cannot speak on the need or lack of need of a "community chest" to take care of the destitute.

Illustrative Problem

Given the frequency table on page 113 of a population of 327 ears of corn. To find the mean deviation of this population of corn from the arithmetic mean length which is 8.83 inches:

Measurement or value, X	Deviation from arithmetic mean, d_a	Frequency, f	Deviation $\times$ frequency, fd_a
3.0	-5.83	1	- 5.83
3.5	-5.33	0	0.00
4.0	-4.83	1	- 4.83
4.5	-4.33	0	0.00
5.0	-3.83	2	- 7.66
5.5	-3.33	3	- 9.99
6.0	-2.83	9	-25.47
6.5	-2.33	8	-18.64
7.0	-1.83	12	-21.96
7.5	-1.33	19	-25.27
8.0	-0.83	32	-26.56
8.5	-0.33	40	-13.20
9.0	0.17	67	11.39
9.5	0.67	63	42.21
10.0	1.17	38	44.46
10.5	1.67	21	35.07
11.0	2.17	8	17.36
11.5	2.67	2	5.34
12.0	3.17	1	3.17
		$\Sigma f = 327$	$\Sigma fd_a ^1 = 318.41$

¹ This symbol means that the numbers are summed as if all were positive. $|fd_a|$ is read, the absolute value of fd_a .

If we divide the sum of the absolute values of the deviations by the total number of ears of corn, we shall have $\frac{\sum |fd_a|}{\sum f} = \frac{318.41}{327} = 0.97$ approximately. This is defined as *mean deviation* of this population of corn from its arithmetic mean.

Mean Deviation.—When the arithmetic mean length, the mode, or the median of a number of measurements is subtracted from each measurement and the sum of their absolute values is divided by the total number of the measurements an *average deviation* is found. An average deviation when computed from the arithmetic mean will be called the *mean deviation*.

Just as the arithmetic mean length ear, 8.83 inches, furnishes one with a measure of the 327 ears considered above (if one were called upon to write a single number that would represent the 327 ears he might write the arithmetic mean), so also does the mean deviation 0.97 furnish one with a measure of the dispersion or variability of the measurements of these ears from the arithmetic mean length.

Standard Deviation.—Another measure of dispersion is used more often than the mean deviation. It is known as the *standard deviation* (S.D.), or the root-mean-square deviation from the arithmetic mean. The formula for standard deviation is $S.D. = \sqrt{\frac{\sum fd_a^2}{\sum f}}$; the usual convention is *sigma*, σ , in place of S.D. In finding the mean deviation illustrated above, we had a column headed "Deviation $\times$ frequency, fd_a ." In finding standard deviation we have a corresponding column, the difference being that the deviations are squared before multiplying by the frequency. The formula tells us to multiply the square of each deviation from the arithmetic mean by its frequency and then add. After dividing the sum of all such numbers by the total number of measurements the square root of the quotient is taken. The formula will be applied in the following problem.

Illustrative Problem

Determine the mean deviation (Mn. D.) and the standard deviation (S.D.), of the numbers that represent the butter-fat percentages given in the frequency table in Exercise 35, Problem 1.

Butter fat, percentage, X	Frequency, f	Deviation from arithmetic mean, ¹ d_a	Frequency $\times$ deviation, fd_a	fd_a^2
5.1	1	1.41	1.41	1.9881
4.5	1	0.81	0.81	0.6561
4.4	1	0.71	0.71	0.5041
4.3	1	0.61	0.61	0.3721
4.2	2	0.51	1.02	0.5202
3.9	2	0.21	0.42	0.0882
3.8	2	0.11	0.22	0.0242
3.7	4	0.01	0.04	0.0004
3.6	1	-0.09	-0.09	0.0081
3.5	1	-0.19	-0.19	0.0361
3.4	4	-0.29	-1.16	0.3364
3.3	5	-0.39	-1.95	0.7605
3.2	2	-0.49	-0.98	0.4802
3.0	1	-0.69	-0.69	0.4761
$\Sigma f = 28$			$\Sigma [fd_a] = 10.30$	$\Sigma fd_a^2 = 6.2508$

$$\text{Mean deviation} = \frac{\Sigma |fd_a|}{\Sigma f} = \frac{10.30}{28} = 0.368.$$

$$\text{Standard deviation} = (\text{deviation from arithmetic mean})$$

$$= \sqrt{\frac{\Sigma fd_a^2}{\Sigma f}} = \sqrt{\frac{6.2508}{28}} = 0.472^2$$

¹ The deviations are measured from arithmetic mean 3.69.

² In computing arithmetic mean, standard deviation, and probable error (to be defined later), the question arises as to how many figures are significant (see KELLEY, T. L., *Science*, Dec. 5, 1924, p. 524.

In comparing the mean deviation and the standard deviation in the above problem notice that in finding the average deviation the column headed fd_a was summed as if each number

were positive while in the corresponding column fd_a^2 in standard deviation each number is necessarily positive as it is the product of a positive number and a number that has been squared. Also notice that by squaring a deviation, its influence is increased if it is large and decreased if it is small.

50. Computation of Standard Deviation by Two Methods.—The guess method used in calculating the arithmetic mean in Art. 48 will now be extended to the computations of standard deviation. For purposes of comparison the standard deviation will be computed by using deviations from the arithmetic mean and by using deviations from a guessed average.

Illustrative Problem

Determine the standard deviation of the numbers that represent pounds of milk given in the illustrative problem in Art. 48.

COMPUTATION OF STANDARD DEVIATION

Long method: deviations from the arithmetic mean 25.37					Short method: deviations from the guessed mean 25.0		
<i>X</i>	<i>f</i>	<i>d_a</i>	<i>d_a</i> ²	<i>fd_a</i> ²	<i>d_g</i>	<i>d_g</i> ²	<i>fd_g</i> ²
23.4	1	-1.97	3.8809	3.8809	-1.6	2.56	2.56
23.8	1	-1.57	2.4649	2.4649	-1.2	1.44	1.44
24.2	3	-1.17	1.3689	4.1067	-0.8	0.64	1.92
24.6	1	-0.77	0.5929	0.5929	-0.4	0.16	0.16
25.0	3	-0.37	0.1369	0.4107	0.0	0.00	0.00
25.4	7	0.03	0.0009	0.0063	0.4	0.16	1.12
25.8	7	0.43	0.1849	1.2943	0.8	0.64	4.48
26.2	2	0.83	0.6889	1.3778	1.2	1.44	2.88
26.6	2	1.23	1.5129	3.0258	1.6	2.56	5.12
27.0	0	1.63	2.6569	0.0000	2.0	4.00	0.00
27.4	1	2.03	4.1209	4.1209	2.4	5.76	5.76
$\Sigma f = 28$				$\Sigma fd_a^2 = 21.2811$			$\Sigma fd_g^2 = 25.44$

By definition, standard deviation, $\sigma = \sqrt{\frac{\Sigma fd_a^2}{\Sigma f}} = \sqrt{\frac{21.2811}{28}} = 0.872$. When deviations have been computed from some quantity *c* distance from the arithmetic mean, then $\sigma = \sqrt{\frac{\Sigma fd_g^2}{\Sigma f} - c^2}$. This formula will

be derived below. In this problem, $c = 25.37 - 25.0 = 0.37$, therefore

$$\sigma = \sqrt{\frac{25.44}{28} - (0.37)^2} = 0.872.$$

$$\text{Derivation of formula } \sigma = \sqrt{\frac{\Sigma f d_o^2}{\Sigma f} - c^2}.$$

Using a and g for arithmetic mean and guessed mean and $a - g = c$. Then $X_1 - a, X_2 - a, X_3 - a, \dots, X_n - a$, and $X_1 - g, X_2 - g, X_3 - g, \dots, X_n - g$ are the respective deviations of the measurements from a and g . (Some of the measurements $X_1, X_2, X_3, \dots, X_n$ may be equal as in the above problem.) Let $S^2 = \frac{\Sigma f d_o^2}{\Sigma f}$ and recall that $\Sigma f = N$, the number of items, then

$$\begin{aligned} N(\sigma^2 - S^2) &= \Sigma f d_a^2 - \Sigma f d_g^2 = (X_1 - a)^2 + (X_2 - a)^2 + (X_3 - a)^2 \\ &\quad \dots + (X_n - a)^2 \\ &\quad - [(X_1 - g)^2 + (X_2 - g)^2 + (X_3 - g)^2 + \dots + (X_n - g)^2] \\ &= -2a(X_1 + X_2 + X_3 + \dots + X_n) + Na^2 \\ &\quad - [-2g(X_1 + X_2 + X_3 + \dots + X_n) + Ng^2] \\ &\quad - 2Na^2 + Na^2 + 2Nga - Ng^2 \text{ since } \frac{X_1 + X_2 + X_3 + \dots + X_n}{N} = a \\ &\quad \text{by definition of arithmetic mean.} \\ &= -N(a^2 - 2ag + g^2) \end{aligned}$$

Therefore

$$\sigma^2 - S^2 = -(a - g)^2 = -c^2$$

and

$$\sigma^2 = S^2 - c^2 = \frac{\Sigma f d_o^2}{\Sigma f} - c^2$$

and

$$\sigma = \sqrt{\frac{\Sigma f d_o^2}{\Sigma f} - c^2}.$$

The following is taken from a tabulated report on "Progression in High and Low Oil Content of 'Illinois' Corn." The complete report is found in the Agricultural Experiment Station *Bulletin* referred to on page 113.

Percentage.....	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00
Frequency.....	1	15	20	41	36	25	15	5	4	1

In this part of the experiment, 163 ears of corn were used and their oil content fell in the different classes as indicated

above. The mean oil content and the standard deviation are given in the following forms: mean = 4.69 ± 0.02 and standard deviation = 0.43 ± 0.02 .

In scientific data it is not sufficient to express the arithmetic mean and the standard deviation as we have been doing. Attached to each with the double sign $\pm$ is what is known as probable error. The probable error¹ of the arithmetic mean of N measurements is defined by the formula $\pm P.E._a$

$$= \pm \frac{\text{standard deviation}}{\sqrt{N}} \times 0.6745. \text{ Similarly the probable}$$

error of the standard deviation of N measurements is defined by the formula $\pm P.E._\sigma = \pm \frac{\text{standard deviation}}{\sqrt{2N}} \times 0.6745.$

We shall now apply these formulas.

Illustrative Problem

Compute the probable error of the arithmetic mean and of the standard deviation of the percentages given in the above frequency table by use of the guessed average.

Number of ears of corn, 163. Guessed average, 5.00 per cent.

Percentage	f	d_g	fd_g	fd_g^2
3.75	1	-1.25	- 1.25	1.5625
4.00	15	-1.00	-15.00	15.0000
4.25	20	-0.75	-15.00	11.2500
4.50	41	-0.50	-20.50	10.2500
4.75	36	-0.25	- 9.00	2.2500
5.00	25	0.00	0.00	0.0000
5.25	15	0.25	3.75	0.9375
5.50	5	0.50	2.50	1.2500
5.75	4	0.75	3.00	2.2500
6.00	1	1.00	1.00	1.0000
$\Sigma f = 163$			$\Sigma fd_g = -50.50$	$\Sigma fd_g^2 = 45.7500$

¹ The meaning of probable error and the significance of the constant 0.6745 as well as a more comprehensive treatment of the material included in this chapter may be found in textbooks on statistics (see GAVETT, "First Course in Statistical Method").

Had our guessed average 5.00 been the true arithmetic mean the algebraic sum of the numbers in the column headed fd_o would have been zero. Since this is not the case we must find the average deviation of our guess from the true mean, which is $-50.50/163 = -0.31$, and correct our guess by this amount. Therefore the true arithmetic mean of the different percentages of oil in the 163 ears is $5.00 - 0.31 = 4.69$ with a probable error to be computed.

By the formula derived on page 125, the standard deviation of these percentages from the arithmetic mean is $\sigma = \sqrt{\frac{45.750}{163} - (-0.31)^2} =$

$0 = 0.43$ with a probable error to be computed. Now that we have the standard deviation we can compute our probable errors by use of the formulas given above. The probable error of the arithmetic mean is $\pm \frac{0.43 \times 0.6745}{\sqrt{163}} = \pm 0.0232 = \pm 0.02$ approximately and the probable

error of the standard deviation is $\pm \frac{0.43 \times 0.6745}{\sqrt{2 \times 163}} = \pm 0.0161 = \pm 0.02$, approximately.

The arithmetic mean and the standard deviation of the percentages of oil in the above 163 ears of corn are: $a = 4.69 \pm 0.02$ and $\sigma = 0.43 \pm 0.02$. Had the experimenter selected from this same population of corn a second random sample, a third, a fourth, etc., and then followed in every detail the experiment referred to above, he would not have expected to obtain the arithmetic mean 4.69; but he would have expected, however, that the different means would fall within the limits 4.67 and 4.71 as often as without in the long run. In general, the *probable error of an average is that value which when added to and subtracted from that average defines limits between which a second average of measurements of a random sample of the same population stands an even chance of falling.*

Exercise 36

1. In the illustrative problem on page 123 the standard deviation of the butter-fat percentages was found to be 0.472. Determine the probable error of the arithmetic mean 3.69. Also find the probable error of the standard deviation.

2. Compute the probable error of the arithmetic mean 25.37 of the pounds of milk given in the illustrative problem in Art. 48. Also compute the probable error of the standard deviation.

3. Under the title "The Ratio of the Length of Petiole to the Length of the Middle Leaflet as a Means of Distinguishing Northern from Southern Alfalfa," a student made a large number of measurements. One of his

tables of data follows. Check his results for the arithmetic mean and probable error.

Locality where grown	Ratio	Locality where grown	Ratio
Argentine.....	3.86	Italy.....	4.27
Argentine.....	4.27	Spain.....	3.53
South Africa.....	3.69	South Africa.....	3.02
Turkestan.....	4.09	South America.....	4.36
Africa.....	3.73	Chubut, South America...	4.30
Argentine.....	4.22	Argentine.....	4.19
Argentine.....	4.35	Argentine.....	4.14
Africa.....	3.40	Argentine.....	4.88
Argentine.....	4.55	South America.....	4.23
South America.....	4.27	Argentine.....	4.34
Turkestan.....	4.56	Turkestan.....	4.00
Chubut, South America.....	4.12	Argentine.....	4.03
Argentine.....	4.69	Turkestan.....	5.32
South America (Pampa).....	4.81	South America.....	4.57

$$\text{Mean} = 4.21 \pm .059$$

51. Coefficient of Variation.—We have found that the arithmetic mean of the butter-fat percentages of the 28 milkings of the Holstein-Friesian cow is 3.69, with a standard deviation of 0.472, and we also found that the arithmetic mean of the pounds of milk for the same 28 milkings is 25.37, with a standard deviation 0.872. We now ask ourselves which is the more variable, the percentage of butter fat or the pounds of milk. In the form that these results are now in, direct comparison is impossible. They can be compared, however, if standard deviation is expressed as a percentage of the arithmetic mean. The coefficient of variability or the coefficient of dispersion is found by dividing the standard deviation by the arithmetic mean. This is expressed as a percentage by the formula: *C. of V.* = $100\delta/\text{Arithmetic Mean}$. For example, the coefficient of variation of the percentage of butter fat is $\frac{0.472}{3.69} = 0.128$ or 12.8 per cent, and

the coefficient of variation of pounds of milk is $\frac{0.872}{25.37} = 0.0344$ or 3.44 per cent. The percentage of butter fat was about three and one-half times more variable than the pounds of milk.

Problem

The arithmetic mean length of the 327 ears of corn in Problem 1, Exercise 33 is 8.83 inches with a standard deviation of 1.28. The arithmetic mean of the oil content of the 163 ears of corn in the illustrative problem, Art. 50, is 4.69 per cent with a standard deviation of 0.43. Which is the more variable, the lengths of the ears in the first population of corn or the oil content in the second population of corn?

Exercise 37

1. The data that follow constitute a portion of a table published in *Bulletin* 138, Michigan State College.

Date	Weights of pigs							
	1	2	3	4	5	6	7	8
Apr. 13.....	12.50	13.25	10.50	12.50	8.75	13.00	7.75	12.25
Apr. 20.....	17.25	18.50	15.75	16.50	12.25	17.75	10.50	17.50
Aug. 10.....	137.50	111.50	118.50	117.00	101.00	123.50	102.00	116.00
Total gain.	125.00	98.25	108.00	104.50	92.25	110.50	94.25	103.75

The data included in the table for weeks intervening from Apr. 20 to Aug. 10 have been omitted.

In *Bulletin* 165 of the University of Illinois the following results concerning these same data are given:

Total gain	Statistical data of total gains in weight		
	Mean	Standard deviation	Coefficient of variation
In 1 week, Apr. 13 to 20.....	4.43	0.86	19.32
In 17 weeks, Apr. 13 to Aug. 10..	104.60	9.77	9.34

- (a) Check mean, standard deviation, and coefficient of variation for week Apr. 13 to 20.
 (b) Check mean, standard deviation, and coefficient of variation for 17 weeks Apr. 13 to Aug. 10.

2. The experiment referred to in Problem 1 also included a litter of nine Poland China pigs with following results:

Date	Weights of pigs								
	1	2	3	4	5	6	7	8	9
Apr. 13.....	3.75	3.63	3.38	5.25	4.50	4.50	5.00	3.00	5.50
Apr. 20.....	6.25	6.25	8.75	7.75	7.50	6.25	8.25	4.75	7.75
Aug. 10.....	83.00	81.50	86.50	91.50	82.50	88.50	95.00	90.50	91.50
Total gain...	79.25	77.87	83.12	86.25	78.00	84.00	90.00	87.50	86.00

The data included in the *Bulletin* for weeks intervening from Apr. 20 to Aug. 10 have been omitted.

- (a) Find the arithmetic mean gain of these nine pigs for the week of Apr. 13 to 20. Find standard deviation and coefficient of variation.
 (b) Answer same questions for the 17 weeks from Apr. 13 to Aug. 10.
 Results given in the *Bulletin* are:

	Mean	Standard deviation	Coefficient of variation
(a).....	2.78	1.04	37.41
(b).....	83.55	4.11	4.92

3. In the experiment referred to in the illustrative problem on page 126 the experimenter gave the following results on the oil-content percentage of corn for the population grown in the year 1903:

Percentage.....	5.50	5.75	6.00	6.25	6.50	6.75	7.00	7.25	7.50	7.75
Number of ears.....	1	10	10	19	18	21	13	5	2	1

Find the mean, the standard deviation, and the coefficient of variability.

NOTE.—The experimenter states that he found his mean before the ears were placed in the respective classes as is done in the above frequency table. His results are: mean = 6.51 ± 0.03 , standard deviation = 0.46 ± 0.02 , coefficient of variability = 7.07 ± 0.34 .

The formula for finding the probable error of the coefficient of variability when variability is below 10 per cent is $\pm P.E. \text{ of } C. \text{ of } V. =$

$$\pm \frac{\text{Coefficient of variability}}{\sqrt{2n}} 0.6745.$$

CHAPTER XV

FUNDAMENTALS OF TRIGONOMETRY

52. Functions of Acute Angles.—We proved in plane geometry that if two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar, and therefore, the corresponding sides are proportional. If in Fig. 7, a perpendicular is dropped from any point in the hypotenuse of the triangle to the base, a second right triangle is formed which is similar to the first and the ratio of any two sides in the second triangle is equal to the ratio of the respective corresponding sides of the first. Since this is true, we shall direct attention for the present to the right triangle in Fig. 7 and to the angle θ .

In the following definitions we speak of the “side opposite” and the “side adjacent” and we mean the side opposite the angle θ and the side adjacent to the angle θ . The six ratios that can be formed from the three sides of the triangle are named sine θ , cosine θ , tangent θ , cotangent θ , secant θ and cosecant θ . These ratios are expressed symbolically as follows:

$$\text{Sin } \theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{y}{r}. \quad (1)$$

$$\text{Cos } \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{x}{r}. \quad (2)$$

$$\text{Tan } \theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{y}{x}. \quad (3)$$

$$\text{Cot } \theta = \frac{1}{\text{tan } \theta} = \frac{x}{y}. \quad (4)$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{r}{x}. \quad (5)$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{r}{y}. \quad (6)$$

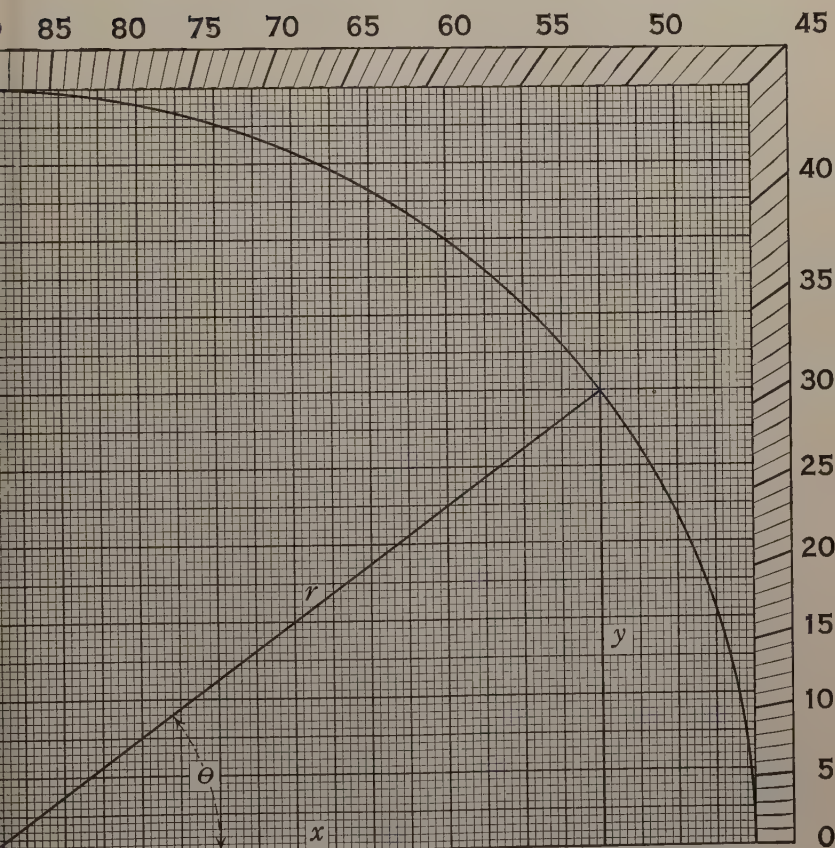


FIG. 7.

The following table gives the values of three trigonometric ratios for intervals of 5° correct to three significant figures:

TABLE A

Angle θ , degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$	Angle θ , degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$
5	0.0872	0.996	0.0875	85	0.996	0.0872	11.4
10	0.174	0.985	0.176	80	0.985	0.174	5.67
15	0.259	0.966	0.268	75	0.966	0.259	3.73
20	0.342	0.940	0.364	70	0.940	0.342	2.75
25	0.423	0.906	0.466	65	0.906	0.423	2.14
30	0.500	0.866	0.577	60	0.866	0.500	1.73
35	0.574	0.819	0.700	55	0.819	0.574	1.43
40	0.643	0.766	0.839	50	0.766	0.643	1.19
45	0.707	0.707	1.00	45	0.707	0.707	1.00

Exercise 38

1. Make a skeleton form of Table A above and by means of Fig. 7 measure sides y , r , and x for each angle included in the first column and compute the respective ratios. Your ratios should be computed to two significant figures and your results should give a good approximation to the numbers in the table. NOTE.—Suppose you wish the sine of 25° . Place a straightedge on the vertex of angle θ and on the marginal check marked 25° . Make a light check where the straightedge cuts the arc of the circle. You now have a means of finding y and r without actually drawing the triangle. Since you are finding a ratio, you are not interested in the length of one of the divisions on the coordinate paper but simply in the numbers that represent y and r . By selecting the triangle of reference as indicated, the hypotenuse remains constant for the different angles θ and is equal to 100 divisions.

2. In any right triangle if one acute angle is θ , then the other acute angle is $(90-\theta)$. Referring to Fig. 7, the sine of the angle $(90-\theta)$ is the ratio of the side opposite this angle, which is x , to the hypotenuse r . But x/r is the cosine of θ , therefore $\sin (90-\theta) = \cos \theta$. (a) Making use of this fact, complete the second half of your table in Problem 1. (b) In Table A why should the ratios under $\sin \theta$ in the first half equal, respectively, the ratios under $\cos \theta$ in the second half?

3. In Problem 2 you have shown that $\sin (90-\theta) = \cos \theta$. Similarly show that $\cos (90-\theta) = \sin \theta$, $\tan (90-\theta) = \cot \theta$, $\cot (90-\theta) = \tan \theta$, $\sec (90-\theta) = \csc \theta$, and $\csc (90-\theta) = \sec \theta$. You have now proved that any trigonometric function of the complement of θ is equal to the cofunction of θ .

4. (a) In Table A we find $\sin 40^\circ = 0.643$ and $\sin 35^\circ = 0.574$, by means of interpolation as was done in using the table of logarithms find $\sin 38^\circ$. Check your result by use of Fig. 7.
- (b) By use of Table A and by means of interpolation find the $\cos 38^\circ$ and check.

NOTE.—When you interpolate notice that as the angle increases the cosine of the angle decreases

(c) Find $\tan 38^\circ$ and check.

5. (a) Construct a right triangle each leg of which is two units in length and by means of the Pythagorean theorem show that the hypotenuse is $2\sqrt{2}$ units in length. From your figure show that $\sin 45^\circ = \frac{1}{2}\sqrt{2}$, $\cos 45^\circ = \frac{1}{2}\sqrt{2}$, and $\tan 45^\circ = 1$.

(b) Construct a right triangle each leg of which is x units long. Show that the hypotenuse is $x\sqrt{2}$ units long. Find the sine, cosine, and tangent of its acute angles.

6. (a) At one end of a horizontal line that is two units long draw a line two units in length and making an angle of 60° with the base line. Form an equilateral triangle by joining the ends of these lines. Drop a perpendicular from the vertex angle to the base. From your figure show that the altitude of the triangle is $\sqrt{3}$ and that $\sin 60^\circ = \frac{1}{2}\sqrt{3}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$, $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{1}{2}\sqrt{3}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$.

(b) Construct an equilateral triangle with sides $2x$ units in length and show that its altitude is $x\sqrt{3}$. Find sine, cosine, and tangent of the 60° angle and of the 30° angle.

7. Place your straightedge on Fig. 7 so that it lies along the hypotenuse of the triangle. What change takes place in the value of the fraction in $\sin \theta = y/r$ as you move the straightedge so as to decrease the angle θ ? As the straightedge moves so as to make angle θ approach 90° , what change takes place in the value of the fraction? Look at the column headed $\sin \theta$ in Table A and see if $\sin \theta$ approaches 0 as θ approaches 0 and if it approaches one as θ approaches 90° . Consider $\cos \theta = x/r$ in a similar manner and then check your conclusions by examining the table. If you consider in like manner $\tan \theta = y/x$, you will find that as θ approaches 90° the fraction increases for the numerator y grows large while the denominator x decreases. When $\theta = 90^\circ$ then x is 0. Since division by 0 is excluded (review the definition of division Art. 1) the tangent of 90° is not defined. Symbolically this is written $\tan 90^\circ = \infty$.

Functions of 0° , 30° , 45° , 60° , and 90° .—If the student has followed the suggestions outlined in Problems 5, 6, and 7, he

has derived the values of the sine, cosine, and tangent of the angles given in the table below. He should not only be able to derive the results in the table but should be able to give the value of each function for the respective angle.

TABLE B

Angle	0°	30°	45°	60°	90°
Sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
Cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
Tan	0	$\frac{1}{2}\sqrt{3}$	1	$\sqrt{3}$	

$$\sqrt{3} = 1.732 \dots, \sqrt{2} = 1.414 \dots$$

The values of the functions for angles 30° , 45° , and 60° as expressed by means of radical signs are not approximations as are those values given in Table A. We know from arithmetic, for example, that $\sqrt{3} = 1.73205 \dots$. If we wish to express $\sin 60^\circ$ as a decimal we shall have $\sin 60^\circ = \frac{1}{2}\sqrt{3} = \frac{1.73205 \dots}{2} = 0.86602 \dots$. In Table A we used the approximation 0.866.

53. Right Triangle.—Let us agree that if we call the sides of a triangle a , b , and c we shall name the angles opposite these sides respectively A , B , and C and if the triangle is a right triangle, angle C will be the right angle.

From the Pythagorean theorem, $a^2 + b^2 = c^2$. In working the exercises that require this equation, use the table of squares and roots given in the appendix, when the numbers are large.

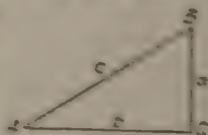


FIG. 5.

Exercise 39

1. Review the definitions of the six trigonometric functions and then work the following. If $a = 3$ and $b = 4$, find c , $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, $\tan A$ and $\cot A$, $\sin B$, $\cos B$, and $\tan B$.

2. Construct a right triangle with $b = 30$ and $a = 16$ and measure side c , angle A , and angle B . Check measurement of c by the Pythagorean theorem. Check angle A as follows: $\tan A = a/b = 16/30 = 0.53$. Refer to Table A and notice that angle A must lie between 25° and 30° . By interpolating as was done in logarithms, angle $A = 28^\circ$.

NOTE. We may use Table A if our protractor reads to degrees only. With half degrees or less and with measured lines of two or three significant figures Table III in appendix should be used.

3. Referring to Fig. 8, suppose BC is a flagpole whose height we wish to find. AC is 86 feet and angle $A = 42^\circ$. By means of Table A, find BC . *Solution:* $BC/86 = \tan 42^\circ = 0.90$, therefore $BC = 0.90 \times 86 = 77$ feet, approximately. Why not use the definition of the $\sin A$ instead of the $\tan A$?

4. Referring to Fig. 8, suppose AC is the distance across a stream, A being a tree on the opposite bank, and that a surveyor finds CB , which lies along the bank, equal to 75 feet and the angle $B = 27^\circ$. Show that the stream is approximately 38 feet in width.

5. In the right triangle ABC angle $A = 60^\circ 0'$, side $b = 47.6$ feet. Show that $c = \frac{b}{\cos A} = \frac{47.6}{\frac{1}{2}} = 95.2$ and that $a = b \tan A = 47.6 \times$

$\sqrt{3} = 82.3$. Substitute the results obtained for a and c in the equation $b = \sqrt{c^2 - a^2}$ and compare the result with the given data.

Solve the following right triangles:

Given:

Required:

6. $b = 16$, $A = 45^\circ$.

$a = 16$, $c = 16\sqrt{2}$, $B = 45^\circ$.

7. $b = 1$, $a = \sqrt{3}$.

$c = 2$, $A = 60^\circ$, $B = 30^\circ$.

8. $a : b : c = 57 : 76 : 95$.

$A = 37^\circ$, $B = 33^\circ$.

9. $a : b : c = \frac{1}{\sqrt{3}} : \frac{1}{3} : \frac{2}{3}$

$A = 60^\circ$, $B = 30^\circ$.

Solution of Right Triangles by Logarithms.—You have proved that $\cos 30^\circ = \frac{1}{2}\sqrt{3}$. When expressed decimally $\cos 30^\circ = \frac{1.73205}{2} = 0.8660 \dots$ We say that the *natural* $\cos 30^\circ = 0.8660$ is correct to four significant figures. To find the *logarithmic cosine* of 30° take the logarithm of each member of $\cos 30^\circ = 0.8660$ and obtain $\log \cos 30^\circ = \log 0.8660 = 9.9375 - 10$ by Art. 20 and Table II, appendix. In solving triangles whose sides have been measured to four or

five significant figures much labor and time is saved by making use of logarithms. Table III, appendix, has been constructed which gives both natural functions and their logarithms. Turn to this table and run down the first column until you come to 30° , then move to the right until you come to column headed "Cosine" and here you will find that the natural $\cos 30^\circ = 0.8660$ and the $\log \cos 30^\circ = 9.9375$ which values are the same, with the exception of the omission of -10 , that were obtained above after much work. The following

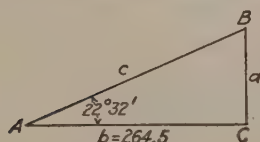


FIG. 9.

problems will illustrate the use of logarithms in solving triangles:

Example 1.—Given $A = 22^\circ 32'$,
 $b = 264.5$.

Find B , a , c .

Draw the figure to scale.

Then write equations that contain but one unknown.

$$\frac{a}{b} = \tan A,$$

therefore

$$a = b \tan A \text{ and } \log a = \log b + \log \tan A \text{ (by}$$

Art. 21, THEOREM 1).

$$= \log 264.5 + \log \tan 22^\circ 32'$$

$$= 2.4224$$

$$+ 9.6179 - 10 \text{ (from Table III, find } \log \tan 22^\circ 30'$$

$$= 2.0403 \text{ and } \log \tan 22^\circ 40' \text{ and interpolate).}$$

$$a = 109.7.$$

$$\frac{b}{c} = \cos A \text{ therefore } c = \frac{b}{\cos A} \text{ and } \log c = \log b - \log \cos A$$

$$= \log 264.5 - \log \cos 22^\circ 32' = 12.4224 - 10$$

$$- 9.9655 - 10$$

$$= 2.4569.$$

Therefore

$$c = 286.3.$$

$$\text{Check: } a^2 = c^2 - b^2 = (c - b)(c + b) = 21.8 \times 550.8.$$

$2 \log a = \log (c - b) + \log (c + b) = \log 21.8 + \log 550.8$. The student should complete this check.

Exercise 40

1. By means of logarithmic functions find the height of a flagpole that stands on a plane, and the angle of elevation of a line drawn from the top of the pole to a point on the ground 216.4 feet from the foot of the pole, is $27^{\circ} 32'$. Make an approximate check by drawing the figure to scale.

2. Given a right triangle ACB with $a = 36.14$, $b = 25.47$. Find A , B , and c by use of logarithms. Apply the check used in the illustrative problem by using your value of c and the given value for b .

3. In the right triangle ACB , the angle $A = 31^{\circ} 28'$ and the hypotenuse $c = 127.4$ ft. Draw the figure to scale and measure a and b . From the figure $\sin A = a/c$. How many unknown quantities in this equation? Find side a by use of logarithms. From $\cos A = b/c$ find b .

4. In the right triangle ABC , the sum of the lengths of its legs is 26.00 and their difference is 6.50. Find the length of the hypotenuse c and the angles A and B .

5. The length of a kite string is 1,156 feet and the angle of elevation of the kite is $36^{\circ} 58'$. If we assume the string to be straight, what is the height of the kite? If a right triangle is formed with the kite at B , the end of the string at A , and the right angle at C , find the length AC . Apply the check used in the illustrative problem.

6. In an isosceles triangle each of the base angles is $36^{\circ} 31'$ and the base is 265.8 feet, find the height of the triangle. Make an approximate check of your work by drawing the figure to scale.

7. A regular hexagon is inscribed in a circle. The distance from the center of the circle to a side of the hexagon is 24.32 feet, what is the radius of the circle?

54. Formulas Connecting the Six Trigonometric Functions.—

Referring to the right triangle ABC (Fig. 8) we have equation (1), $a^2 + b^2 = c^2$. Dividing each member of equation (1) by c^2 and using index Law 5 (Art. 17), we have $(a/c)^2 + (b/c)^2 = 1$. Substituting $\sin A = a/c$ and $\cos A = b/c$ we obtain $(\sin A)^2 + (\cos A)^2 = 1$. We shall agree to write $\sin^2 A$ for $(\sin A)^2$ and $\cos^2 A$ for $(\cos A)^2$ then our equation takes the form:

$$\sin^2 A + \cos^2 A = 1. \quad (2)$$

Similarly

$$1 + \tan^2 A = \sec^2 A \quad (3)$$

and

$$1 + \cot^2 A = \csc^2 A. \quad (4)$$

Equation (3) is obtained by dividing equation (1) by b^2 and equation (4) is obtained by dividing equation (1) by a^2 and then using the definitions of $\tan A$, $\sec A$, $\cot A$ and $\csc A$. Another useful formula is

$$\tan A = \frac{\sin A}{\cos A} \quad (5)$$

which is easily verified by referring to Fig. 8.

In deriving these formulas we used the equation $a^2 + b^2 = c^2$ and we know by the Pythagorean theorem that this equation is satisfied if angle A is any acute angle. Because of this fact we call such formulas as equations (2), (3), (4), and (5) **trigonometric identities**. We shall see the difference between these identities and the trigonometric equations that are considered in the next section. By means of the above identities and the definitions of the six trigonometric ratios we are able to find algebraically the remaining functions when given one of them and we can derive new formulas.

Exercise 41

Prove the following:

1. $\sin A = \pm \sqrt{1 - \cos^2 A}$.

Suggestion: Solve identity (2) for $\sin A$.

2. $\tan A = \frac{\sqrt{1 - \cos^2 A}}{\cos A}$.

Use identity (5) and Problem 1.

3. $\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$.

4. (a) $\sin A \csc A = 1$, (b) $\tan A \cot A = 1$, (c) $\cos A \sec A = 1$.

5. $\cos A + \tan A \sin A = \sec A$.

Solution.— $\cos A + \frac{\sin A}{\cos A} \sin A = \sec A$ by use of identity (5), then

$$\frac{\cos^2 A + \sin^2 A}{\cos A} = \sec A \text{ by simplifying first member}$$

and

$$\frac{1}{\cos A} = \sec A \text{ by identity (2)}$$

and

$$\sec A = \sec A \text{ by definition of secant.}$$

6. Given $\cos A = \frac{5}{13}$, find algebraically the remaining five functions of angle A .

Solution.—Substituting this value of $\cos A$ in the formula given in Problem 1 we have:

$$(a) \sin A = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}.$$

In the following exercises we shall assume that angle A is acute and, therefore, we shall use the positive sign. The use of the negative sign will be explained later in connection with obtuse angles. (See Art. 56.)

$$\begin{aligned} (b) \tan A &= \frac{\sin A}{\cos A} \text{ by identity (5)} \\ &= \frac{\frac{12}{13}}{\frac{5}{13}} \text{ by substitution} \\ &= \frac{12}{5}. \end{aligned}$$

(c) Since $\csc A$, $\sec A$, and $\cot A$ are, respectively, the reciprocals of $\sin A$, $\cos A$, and $\tan A$ we know their values immediately.

Check: Draw a right triangle ABC with $b = 5$, $c = 13$ and $a = \sqrt{c^2 - b^2} = \sqrt{144} = 12$ and then by the definitions of the six functions determine their values by referring to the figure.

7. In the following the value of one function of A is given. Find the values of the remaining functions of A . Use Problem 6 as a guide and assume angle A acute.

$$(a) \sin A = \frac{4}{5}.$$

$$(d) \cot A = \frac{3}{4}.$$

$$(b) \cos A = \frac{1}{2}.$$

$$(e) \sec A = \frac{5}{4}.$$

$$(c) \tan A = \frac{3}{4}.$$

$$(f) \csc A = \frac{5}{3}.$$

8. Given $\cos A = \frac{1}{2}\sqrt{3}$, find the numerical values of the remaining five functions.

55. Trigonometric Equations.—We have seen that the identities considered in Art. 54 are equations that are satisfied for any acute angle for which the trigonometric functions involved are defined. The equations that we shall now consider are satisfied for a limited number of values of the acute angle. To illustrate:

Example 1.—Find all the positive acute angles A that will satisfy the equation (a) $2 \cos^2 A - 3 \cos A + 1 = 0$. Let $\cos A = x$ then equation (a) becomes

$$2x^2 - 3x + 1 = 0$$

factoring we get

$$(2x - 1)(x - 1) = 0,$$

and

$$x = \frac{1}{2} \text{ and } x = 1.$$

Therefore, $\cos A = \frac{1}{2}$, $A = 60^\circ$ (Table B) and $\cos A = 1$, $A = 0^\circ$. Notice that equation (a) could have been solved by factoring it as it stands and considering $\cos A$ as the unknown. The factored form is $(2 \cos A - 1)(\cos A - 1) = 0$, and when solved $\cos A = \frac{1}{2}$, $A = 60^\circ$, and $\cos A = 1$, $A = 0^\circ$.

Example 2.—Find all the positive acute angles A that will satisfy the equation (b) $\cos A + \sec A = \frac{5}{2}$. The first step is to transform it to an equation in a single function, which can be done by substituting $\sec A = \frac{1}{\cos A}$. Equation (b) then becomes after being simplified $2 \cos^2 A - 5 \cos A + 2 = 0$. Factoring we get

$$(2 \cos A - 1)(\cos A - 2) = 0$$

from which

$$2 \cos A - 1 = 0, \text{ or } \cos A = \frac{1}{2}, A = 60^\circ$$

$$\cos A - 2 = 0, \text{ or } \cos A = 2,$$

since $\cos A = b/c$ (Fig. 8) the largest value that $\cos A$ can have is unity and therefore an angle A does not exist for which $\cos A = 2$

Exercise 42

1. Find those angles, 90° or less, that satisfy the following equations:

(a) $\sin^2 A = \cos^2 A$.

(b) $4 \cos A = 3 \sec A$.

(c) $2 \cos^2 A + 5 \sin A = 4$.

(d) $3 \sin A - 2 \cos^2 A = 0$.

(e) $\sin A + \csc A = 2$.

2. Find the value of B less than 90° that satisfies the equation $6 \cos B \cot B = 5$.

56. Trigonometric Functions of Any Angle.—As in plane geometry, we shall think of an angle as generated by holding one of its sides (initial side) fast while the other side (terminal side) rotates about the vertex. If an angle is placed on a pair of axes so that the vertex coincides with the origin and the initial side lies along the positive end of the x -axis and if the terminal side lies in the first quadrant, the angle is said to lie in the first quadrant. If the terminal line falls in the second quadrant, however, the angle is said to lie in the second quadrant; similarly for the third and fourth quadrants. If the terminal side rotates from the initial side counterclockwise, we shall agree to call the angle positive. If the rotation is clockwise, the angle will be negative. With this agreement as to manner of generating an angle we see that an angle can have any size and be positive or negative. For example, angle θ might equal negative 375° or positive 1842° . Suppose

an angle has been placed on rectangular axes as described, and for convenience of discussion, as the terminal line makes one complete rotation we trace the path (circle) of some fixed point P on this line as in Fig. 10. Let us fix our attention on the acute angle XOP_1 , on the obtuse angle XOP_2 , and on the two reflex angles XOP_3 and XOP_4 .

We shall define the trigonometric functions of these angles, or of angles formed by adding any multiple of 360° to each of these angles as follows:

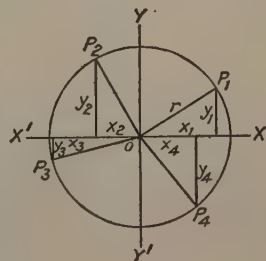


FIG. 10.

$$\sin XOP = \frac{y}{r} \qquad \csc XOP = \frac{1}{\sin XOP} = \frac{r}{y} \text{ (if } y \neq 0 \text{)}$$

$$\cos XOP = \frac{x}{r} \qquad \sec XOP = \frac{1}{\cos XOP} = \frac{r}{x} \text{ (if } x \neq 0 \text{)}$$

$$\tan XOP = \frac{y}{x} \text{ (if } x \neq 0 \text{)} \qquad \cot XOP = \frac{1}{\tan XOP} = \frac{x}{y} \text{ (if } y \neq 0 \text{)}$$

where P , x , y , take the subscript 1, 2, 3, or 4 as in the figure and x , y take the algebraic signs according to the convention of Art. 4. We shall further agree that r will always be positive.

NOTE.—Whenever an angle is such that the denominator of the ratio of the function is zero the function is not defined for that angle; for example, $\tan XOP$ is not defined when angle $XOP_1 = 90^\circ$ (see Art. 1, assumption 6). Each of the trigonometric functions will take an algebraic sign according to the agreement in Art. 4, that is, x is positive if it lies along OX , negative if it lies along OX' and y is positive if it lies above the x -axis and is negative if it lies below the x -axis.

Illustrative Problem

Determine the algebraic sign of $\tan XOP = y/x$ (Fig. 10) for each of the four quadrants. In the first quadrant the sign is positive since the sign of both y and x are positive. In the second quadrant the sign is negative since the sign of y is positive and of x is negative. In the third quadrant the sign is positive since the sign of both y and x are negative. In the fourth quadrant the sign is negative since the sign of y is negative and the sign of x is positive.

Exercise 43

1. Follow the method used in the above illustrative problem and verify and complete the following table which gives the algebraic signs of the functions of angle $XOP = \theta$ when the angle θ lies in the different quadrants:

Quadrant	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
First.....	+	+	+			
Second.....	+	—	—			
Third.....	—	—	+			
Fourth.....	—	+	—			

NOTE.—When completing the table use the fact that the reciprocal of a function has the same algebraic sign as the function. Notice also that the algebraic sign of $\sin \theta$ is positive in the first and second quadrants, $\cos \theta$ is positive in the first and fourth quadrants, and the $\tan \theta$ is positive in the first and third quadrants. With this in mind the remainder of the table can be reproduced at sight.

2. Determine the algebraic sign of each of the following functions by placing the angle on rectangular axes:

(a) $\sin 210^\circ$.

Since angle 210° when placed on the axes lies in the third quadrant $\sin 210^\circ$ is negative.

(b) $\csc 210^\circ$.

(c) $\cos 179^\circ$.

(d) $\sec 179^\circ$.

(e) $\tan 46^\circ$.

(f) $\cot 46^\circ$.

(g) $\tan 340^\circ$.

(h) $\cot 340^\circ$.

(i) $\cos (-30^\circ)$.

(j) $\sin (-45^\circ)$.

(k) $\tan (-60^\circ)$.

3. Verify the following table which gives the values of angle $XOP = \theta$ (Fig. 10) when this angle takes the values: 0° , 90° , 180° , 270° , and 360° . The blank spaces in the table indicate that the function is not defined for that angle.

Angle	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
0°	0	1	0	..	1	
90°	1	0	..	0	...	1
180°	0	-1	0	..	-1	
270°	-1	0	..	0	...	-1
360°	0	1	0	..	1	

NOTE.—See Exercise 38, problem 7 for suggestions as to method for working this problem.

4. (a) Find $\sin 120^\circ$, $\cos 120^\circ$, and $\tan 120^\circ$.

Solution.—Place the 120° angle on the rectangular axes and from a point on the terminal line r distance from the origin, drop a perpendicular to the x -axis and designate it by y (Fig. 11). By Art. 56,

$$\sin 120^\circ = \frac{y}{r}$$

$$\cos 120^\circ = \frac{x}{r}$$

and

$$\tan 120^\circ = \frac{y}{x}$$

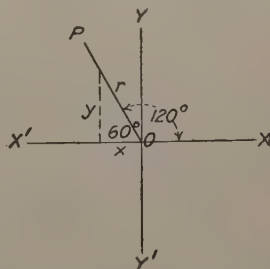


FIG. 11.

If r is two units long, then from Problem 6a, Exercise 36, $y = \sqrt{3}$, $x = -1$ (the minus sign is attached since x lies along OX'). By substituting

these values in the above formulas, $\sin 120^\circ = \sqrt{3}/2$, $\cos 120^\circ = -1/2$, and $\tan 120^\circ = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. (Is it necessary to take r two units long?)

NOTE.—This problem can be worked by determining the sign of the three functions immediately from the fact that the angle lies in the second quadrant, and therefore, the sine is positive, the cosine is negative, and the tangent negative. Then notice that

$$\frac{y}{r} = \sin 60^\circ = \frac{1}{2}\sqrt{3}, \frac{x}{r} = \cos 60^\circ = \frac{1}{2}, \text{ and } \frac{y}{x} = \tan 60^\circ = \sqrt{3}.$$

- (b) In Fig. 11 consider angle XOP as any angle greater than 90° and less than 180° and express $\sin XOP$, $\cos XOP$, and $\tan XOP$ as functions of the acute angle $X'OP$.

Solution.—For convenience replace angle $X'OP$ by ϕ then angle $XOP = (180^\circ - \phi)$ and $\sin (180^\circ - \phi) = \sin \phi$, $\cos (180^\circ - \phi) = -\cos \phi$ and $\tan (180^\circ - \phi) = -\tan \phi$. Since the terminal side of angle $(180^\circ - \phi)$ falls in the second quadrant the algebraic signs of these three functions are respectively, plus, minus, and minus. Since x , the adjacent side, and y , the side opposite the acute angle ϕ , are identical to the x and y used in the definitions of the functions of the obtuse angle $(180^\circ - \phi)$, each of these functions of the obtuse angle are equal in numerical value to the same function of the acute angle.

- (c) In Fig. 11 suppose the terminal line OP rotated so that angle XOP is greater than 180° and less than 270° (notice that angle $XOP = (180^\circ + \phi)$).

Show that $\sin (180^\circ + \phi) = -\sin \phi$, $\cos (180^\circ + \phi) = -\cos \phi$ and $\tan (180^\circ + \phi) = \tan \phi$.

- (d) Rotate the terminal line OP (Fig. 11) so that the angle XOP is greater than 270° and less than 360° and name the acute angle formed by OP and the positive end of the x -axis, ϕ as in Fig. 12 then the reflex angle $XOP = (360^\circ - \phi)$.

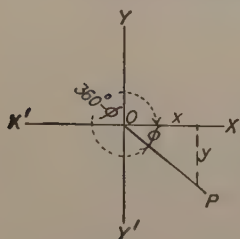


FIG. 12.

Show that $\sin (360^\circ - \phi) = -\sin \phi$, $\cos (360^\circ - \phi) = \cos \phi$ and $\tan (360^\circ - \phi) = -\tan \phi$.

- (e) By means of a figure show that $\sin (-\phi) = -\sin \phi$, $\cos (-\phi) = \cos \phi$, $\tan (-\phi) = -\tan \phi$ and that any one of the trigonometric functions of $(360^\circ + \phi)$ is equal

to the same function of ϕ with the proper sign.

5. Determine the value of each of the following functions:

- (a) $\cos 170^\circ$. *Solution:* Place angle 170° on rectangular axes. The terminal side falls in the second quadrant and therefore the sign

of the function is negative. The acute angle between the terminal line and the x -axis is 10° and since the numerical value of $\cos 10^\circ$ is the same as that of $\cos 170^\circ$ (the student should show this by use of figure), look in Table III, appendix, for $\cos 10^\circ$ and find 0.9848. Therefore $\cos 170^\circ = -0.9848$.

(b) $\sin 170^\circ, \tan 170^\circ$.

(c) $\sin 300^\circ = -\sin 60^\circ = -0.8660$.

(d) $\cos 300^\circ, \tan 300^\circ, \cot 300^\circ$.

(e) $\tan 190^\circ = \tan 10^\circ = 0.1763, \sin 190^\circ = -\sin 10^\circ = 0.1736, \cos 190^\circ, \cot 190^\circ$.

(f) $\sin 120^\circ, \cos 120^\circ, \tan 120^\circ, \cot 120^\circ$.

(g) $\sin (-30^\circ) = -\sin 30^\circ = -0.5000, \cos (-30^\circ), \tan (-30^\circ)$.

Functions of $(-\phi)$, $(180^\circ \pm \phi)$ and $(360^\circ \pm \phi)$.—By means of the above exercises we see that functions of large angles can be reduced to functions of angles less than 90° . A rule for such a reduction may be stated.

The trigonometric functions of $(-\phi)$, $(180^\circ \pm \phi)$, and of $(360^\circ \pm \phi)$ are equal in numerical value to the same functions of ϕ . The algebraic sign attached to the function of the acute angle is the same as the sign of the given function which sign depends on the quadrant in which the angle lies.

NOTE.—In problem 4, ϕ is an acute angle; the conclusions would have been the same had ϕ been any angle for which the function is defined.

Functions of $(90^\circ \pm \phi)$ and $(270^\circ \pm \phi)$.—In the above problems we directed attention to the acute angle ϕ that was formed by the terminal line OP and the x -axis. We shall now direct attention to the acute angle formed by the terminal line OP and the y -axis and call this angle ϕ . A rule for a reduction of angles may be stated:

The trigonometric functions of $(90^\circ \pm \phi)$ and of $(270^\circ \pm \phi)$ are equal in numerical value to the cofunctions of ϕ . The algebraic sign attached to the function of the acute angle is the same as the sign of the given function.

This rule will give the following: $\sin (90^\circ + \phi) = \cos \phi$, $\cos (90^\circ + \phi) = -\sin \phi$, $\tan (90^\circ + \phi) = -\cot \phi$, $\sin (270^\circ -$

$\phi) = -\cos \phi$, $\cos (270^\circ - \phi) = -\sin \phi$, $\tan (270^\circ - \phi) = \cot \phi$, $\sin (270^\circ + \phi) = -\cos \phi$, $\cos (270^\circ + \phi) = \sin \phi$, $\tan (270^\circ + \phi) = -\cot \phi$.

By means of figures the student should satisfy himself that these reductions from large angles to acute angles are correct.

Exercise 44

1. By means of the above rules check the following:

- (a) $\cos 160^\circ = \cos (180^\circ - 20^\circ) = -\cos 20^\circ$. Place angle 160° on the coordinate axes and give reason for the negative sign used in final result.
- (b) $\cos 160^\circ = \cos (90^\circ + 70^\circ) = -\sin 70^\circ$. From the figure show that the results of (a) and (b) are equivalent.
- (c) $\tan 345^\circ = \tan (360^\circ - 15^\circ) = -\tan 15^\circ$.
- (d) $\tan 345^\circ = \tan (270^\circ + 75^\circ) = -\tan 75^\circ$.
- (e) $\sin (-35^\circ) = -\sin 35^\circ$.
- (f) $\cos (-35^\circ) = \cos 35^\circ$.

2. A man travels west 1,764 yards and then travels for 997 yards on a road that has a bearing N. $38^\circ 15'$ W. (the angle between north-and-south line and the road is $38^\circ 15'$). How far west is he from the starting point?

3. Draw to scale a line AB of length 26.7 and then construct an angle BAC of $192^\circ 30'$. Show that $\cos 192^\circ 30' = -\cos 12^\circ 30'$ and compare the graphical result with the computed result.

4. A surveyor uses the east-and-west line as x -axis with east as positive and the north-and-south line as y -axis with north as positive and station

A as origin. He then finds the coordinates of different stations to be: $A (0, 0)$, $B (176.4, 0)$, $C (276.5, 276.5)$. At station C he lays off an angle E. $103^\circ 30' N$. (look east and then turn through this angle) and then chains a distance of 327.4 to station D . Find how far north station D is of station A . Draw an approximate figure.

The exercise that follows is preliminary to the theory in the next section.

5. In the accompanying figure angles BAO , BQR , and PBO are right.

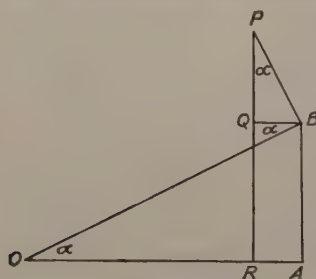


FIG. 13.

- (a) If angle $AOB = \alpha$, show that angle $QBO = \alpha =$ angle QPB .
- (b) Why is $QP = PB \cos \alpha$, $RQ = AB = OB \sin \alpha$, $RA = QB = PB \sin \alpha$, and $OA = OB \cos \alpha$?

- (c) Use the results in parts (a) and (b) and show that $RP = RQ + QP$
 $= OB \sin \alpha + PB \cos \alpha$; $OR = OA - QB = OB \cos \alpha - PB \sin \alpha$.
- (d) Join P to O and let the angle $BOP = \beta$, also recall that PBO is a right angle. Then $OB/OP =$ what? $PB/OP =$ what?
- (e) Is angle $ROP = \alpha + \beta$?
- (f) $\sin ROP =$ what? $\sin (\alpha + \beta) =$ what? $\cos (\alpha + \beta) =$ what?

57. Functions of the Sum and of the Difference of Two Angles.—We shall prove that:

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

Before proving the formulas we shall apply them to some simple exercises. If $\alpha = 30^\circ$ and $\beta = 60^\circ$ then by substituting these values in the first formula we have

$$\begin{aligned} \sin (30^\circ + 60^\circ) &= \sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ \\ &= \frac{1}{2} \cdot \frac{1}{2} & \quad \quad \quad + \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\sqrt{3} \\ &= \frac{1}{4} & \quad \quad \quad + \frac{3}{4}. \end{aligned}$$

Therefore, $\sin 90^\circ = 1$ which we know to be true. Similarly, by substituting in the second formula we have

$$\begin{aligned} \cos (30^\circ + 60^\circ) &= \cos 30^\circ \cos 60^\circ - \sin 30^\circ \sin 60^\circ \\ &= \frac{1}{2}\sqrt{3} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2}\sqrt{3}. \end{aligned}$$

Therefore, $\cos 90^\circ = 0$ which we know to be true. We shall prove the above formulas when angles α and β are positive acute angles whose sum is less than 90° . The formulas are true, however, for values of α and β positive or negative and of any magnitude whatever.

In Fig. 14, angle AOP is the sum of angles α and β and RP is perpendicular to OA .

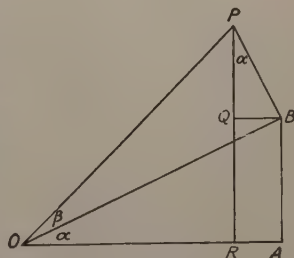


FIG. 14.

$$\begin{aligned}(a) \sin(\alpha + \beta) &= \frac{RP}{OP} = \frac{OB}{OP} \sin \alpha + \frac{PB}{OP} \cos \alpha. \\ &= \cos \beta \sin \alpha + \sin \beta \cos \alpha.\end{aligned}$$

See Problem 5 above for details.

$$\begin{aligned}(b) \cos(\alpha + \beta) &= \frac{OR}{OP} = \frac{OB}{OP} \cos \alpha - \frac{PB}{OP} \sin \alpha. \\ &= \cos \beta \cos \alpha - \sin \beta \sin \alpha.\end{aligned}$$

(c) In (a) replace β by $-\beta$ and recall that $\sin(-\beta) = -\sin \beta$, and $\cos(-\beta) = \cos \beta$, then $\sin(\alpha - \beta) = \cos(-\beta) \sin \alpha + \sin(-\beta) \cos \alpha = \cos \beta \sin \alpha - \sin \beta \cos \alpha$.

(d) Similarly (b) becomes

$$\begin{aligned}\cos(\alpha - \beta) &= \cos(-\beta) \cos \alpha - \sin(-\beta) \sin \alpha \\ &= \cos \beta \cos \alpha + \sin \beta \sin \alpha\end{aligned}$$

Exercise 45

1. (a) In the formula for $\sin(\alpha + \beta)$ substitute $\alpha = 180^\circ$ and assume that β is any positive acute angle and show that $\sin(180^\circ + \beta) = -\sin \beta$, $\sin(180^\circ - \beta) = \sin \beta$.
- (b) If $\alpha = 90^\circ$, show that $\sin(90^\circ + \beta) = \cos \beta$, $\sin(90^\circ - \beta) = \cos \beta$.
- (c) In the formula for $\cos(\alpha + \beta)$ substitute $\alpha = 180^\circ$ and show that $\cos(180^\circ + \beta) = -\cos \beta$, $\cos(180^\circ - \beta) = -\cos \beta$.
- (d) If $\alpha = 90^\circ$, show that $\cos(90^\circ + \beta) = -\sin \beta$, $\cos(90^\circ - \beta) = \sin \beta$.

2. By identity 5, Art. 54, $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$. Give a reason for each step in what follows:

$$\begin{aligned}(a) \tan(\alpha + \beta) &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}.\end{aligned}$$

$$\text{Therefore, } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

$$\begin{aligned}
 (b) \tan (45^\circ + 30^\circ) &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\
 &= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} \\
 &= \frac{\left(1 + \frac{\sqrt{3}}{3}\right)^2}{1 - \frac{1}{3}} \\
 &= \frac{1 + \frac{2}{3}\sqrt{3} + \frac{1}{3}}{\frac{2}{3}} = 2 + \sqrt{3}.
 \end{aligned}$$

Therefore, $\tan 75^\circ = 3.73$.

3. (a) By replacing β by $-\beta$ in the formula in Problem 2, show that

$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

(b) Find a numerical value for $\tan 15^\circ$ by using part (a) and Table B.
Check your result by use of Table A.

4. Use Problems 2 and 3 and show that

$$(a) \tan (180^\circ + \beta) = \tan \beta; \quad (b) \tan (180^\circ - \beta) = -\tan \beta.$$

5. Use identity 5, Art. 54, and Problem 1, and show that

$$(a) \tan (90^\circ + \beta) = \frac{\sin (90^\circ + \beta)}{\cos (90^\circ + \beta)} = \frac{\cos \beta}{-\sin \beta} = -\cot \beta.$$

$$(b) \tan (90^\circ - \beta) = \text{what?}$$

6. Given that $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$,
and $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.

Show that

$$(a) \sin 2\alpha = 2 \sin \alpha \cos \alpha;$$

$$(b) \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1.$$

Hint: Replace β by α .

7. Given that $\sin A = \frac{1}{3}$ and that angle A is acute. Find $\cos A$, $\sin 2A$, and $\cos 2A$.

8. From Problem 2, $\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$.

Show that

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

9. From Problem 6, $\cos 2A = 1 - 2 \sin^2 A$, show that

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

and that

$$\sin A = \pm \sqrt{\frac{1 - \cos 2A}{2}}.$$

In this formula let $A = \frac{\alpha}{2}$ and obtain $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$

10. From Problem 6, $\cos 2A = 2 \cos^2 A - 1$, show that

$$\cos^2 A = \frac{1 + \cos 2A}{2};$$

that

$$\cos A = \pm \sqrt{\frac{1 + \cos 2A}{2}};$$

and

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \text{ when } 2A = \alpha.$$

11. By use of the formula in Problem 10, find $\cos 15^\circ$, having given that $\cos 30^\circ = \frac{1}{2}\sqrt{3} = \frac{1.732}{2} = 0.866$ (see Table B).

$$\text{Solution.}—\cos 15^\circ = \left(\frac{1 + \cos 30^\circ}{2}\right)^{\frac{1}{2}} = \left(\frac{1.866}{2}\right)^{\frac{1}{2}} = (0.933)^{\frac{1}{2}}$$

$$\log \cos 15^\circ = \frac{1}{2} \log 0.933 = \text{etc.}$$

12. (a) Show that $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$, and that

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}.$$

(b) Show that $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta$, and that

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}.$$

$$\begin{aligned} \text{(c) Show that } \frac{\sin A + \sin B}{\sin A - \sin B} &= \tan \frac{A+B}{2} \cot \frac{A-B}{2}, \\ &= \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}. \end{aligned}$$

58. **Circular Measure of Angles.**—A central angle that stands on $\frac{1}{180}$ of a semicircumference is said to have a measure

of 1° and a central angle such as AOB that stands on an arc equal to the radius r of the circle is said to have a measure of one **radian**. Since from plane geometry we know that in the same circle central angles are proportional to the arcs on which they stand, therefore, from Fig. 15:

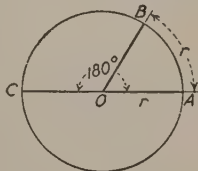


FIG. 15.

$$\text{Angle } AOB : \text{arc } AB = \text{angle } AOC : \text{arc } \pi r,$$

or

$$\text{one radian} : r = 180^\circ : \pi r$$

or

$$\pi \text{ radians} = 180^\circ.$$

This equation is fundamental as it connects angles measured in radians with angles measured in degrees. For example:

$$\frac{\pi}{2} \text{ radians} = 90^\circ,$$

$$\frac{\pi}{3} \text{ radians} = 60^\circ,$$

$$\frac{\pi}{4} \text{ radians} = 45^\circ,$$

or

$$180^\circ = \pi \text{ radians},$$

$$1^\circ = \frac{3.1416}{180} \text{ radians},$$

$$\begin{aligned} 30^\circ &= \frac{3.1416}{6} \text{ radians}, \\ &= 0.5236 \text{ radians}. \end{aligned}$$

The student should drill himself in radian measure so that, for example, $\pi/6$ radians suggest immediately 30° . In connection with the trigonometric functions the word *radian* is omitted. We do not write $\sin \pi/6$ radians but $\sin \pi/6$.

Exercise 46

1. From Table B, find the values for $\sin \pi/6$, $\tan \pi/4$, $\cos \pi/2$, $\sec \pi/3$.
2. How many degrees in the following angles: 3π radians, $\frac{3}{2}$ radian, $\frac{1}{12}\pi$ radians, $\frac{2}{3}\pi$ radians, $\frac{3}{4}\pi$ radians.

3. Find the length of an arc subtending an angle (a) of one radian at the center of a circle of 12 feet, (b) of 2.5 radians at the center of a circle of 16 feet.

4. Find the length of the radius of a circle at whose center an angle of 3 radians is subtended by an arc of 15 feet.

5. (a) $\sin^2 \pi/3 + \cos^2 \pi/6 = \text{what?}$ (b) $\cot \pi/4 + \csc \pi/3 = \text{what?}$

6. Change to radians: $5^\circ 30'$, 135° , 270° , $1^\circ 30' 30''$.

7. Use the scale of 20 small divisions equal one unit and draw a circle on coordinate paper with radius

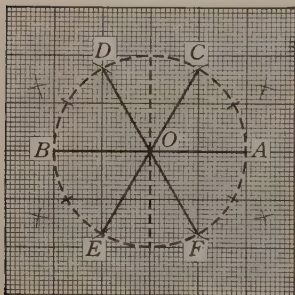


FIG. 16.

$r = 1$. Letter the horizontal diameter from right to left AOB (Fig. 16) and with the same radius and centers A and B strike arcs cutting the circumference at C and at D . Draw diameters CE and DF .

(a) Show that the circumference is divided into six equal arcs.

(b) Bisect the arcs AC , CD , and DB and draw three diameters each having one of these points as an end point.

(c) The semicircumference is now divided into six equal divisions. Show that the lengths of the arcs begin-

ning with A and extending, respectively, to these points of division are $\pi/6$, $2\pi/6$, etc.

(d) Place this circle on the line BOA extended, so that point A on circle coincides with point A on line. If the circle is rolled, without slipping, along the line BOA extended, show that B will fall at a distance from A approximately equal to $3.14 \times 20 = 63$ small divisions and that the equal points of division will fall approximately at the $10\frac{1}{2}$, 21, $31\frac{1}{2}$, etc., divisions on the line.

(e) Since the circle has a radius $r = 1$, show that the $\sin \pi/6$ has the same measure as the length of the perpendicular dropped from the point marked $\pi/6$ to the diameter BOA .

(f) Drop a perpendicular from each point of equal division on circumference to the diameter AOB and compare their respective measures with the numerical values of $\sin \pi/6$, $\sin \pi/3$, $\sin \pi/2$, etc.

(g) At the six points of divisions on the extension of the diameter BOA found as in part (d) erect perpendiculars equal respectively to the perpendiculars in part (f) and join the end points by a smooth curve. Compare your figure with Fig. 17.

59. Variation of the Trigonometric Functions.—The circle in Fig. 17 has a unit radius. The length of its circumference equals the distance from the point A to the point marked 2π on the diameter BOA extended and the 12 equal parts into which the circumference of the circle is divided are indicated. (See Problem 7, page 154.) Since the radius of the circle is unity the perpendiculars dropped from the points marked $\pi/6$, $2\pi/6$, etc., have the same measure as $\sin \pi/6$, $\sin 2\pi/6$, etc. These perpendiculars have been transferred to the corresponding points on the base line to the right of A and a smooth curve drawn through their end points. We shall refer to this curve as a **sine curve**. It is evident that any perpendicular dropped from this curve on the base line has the same measure as the sine of the angle indicated by the abscissa of its foot. For example, the abscissa of the point halfway between $\pi/6$ and $2\pi/6$ is $\pi/4$, and the measure of the perpendicular drawn from the curve to this point is equal to $\sin \pi/4$, or graphically, the perpendicular dropped from the point of bisection of the quadrant arc of the circle is equal to the perpendicular dropped from the curve to the point $\pi/4$ on the base line.

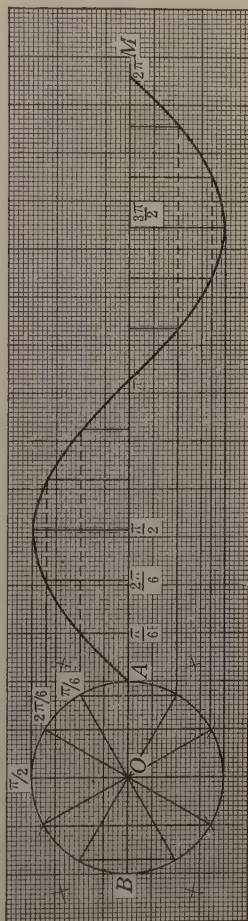


FIG. 17.

The cosine curve may be drawn on the same diagram as follows: Transfer the perpendicular dropped from the point $\pi/2$ on the circumference to the point A on the base line; similarly transfer the perpendicular dropped from the point $2\pi/6$ on the circumference to the point $\pi/6$ on the base line, etc. This construction will lead to the cosine curve since the $\sin \pi/2 = \cos 0$, $\sin 2\pi/6 = \cos \pi/6$, etc.

Exercise 47

The variations of the trigonometric functions may be studied by graphing them as follows: Choose axes as was done in graphing algebraic functions. On the x -axis lay off 0° , 10° , 20° , 30° , 40° . . . 360° to a convenient scale and on the y -axis use a scale such that you can lay off the corresponding value of the function.

(a) Graph $y = \sin x$.

NOTE.—Cut back data to two significant figures.

(b) On same axes and to the same scale graph, $y = \cos x$.

(c) Find the graph of $y = \sin x + \cos x$ by adding the ordinates of part (b) to the corresponding ordinates of part (a).

(d) Graph $y = \tan x$.

60. Inverse Functions.—Given the angle 30° , what is its sine? The answer is, $\sin 30^\circ = \frac{1}{2}$. The inverse operation is, given that the sine of an angle is $\frac{1}{2}$, what is the angle? By use of Fig. 17 or Fig. 10 it is easily shown that the angle may be 30° , 150° , $360^\circ + 30^\circ$, $360^\circ + 150^\circ$, -330° , etc. The symbol $\sin^{-1}$ is used to denote an angle whose sine is whatever number that follows. For example $\sin^{-1} \frac{1}{2}$ denotes an angle whose sine is $\frac{1}{2}$ and is read inverse sine $\frac{1}{2}$. This might be written $\theta = \sin^{-1} \frac{1}{2}$, and is read, θ is an angle whose sine is $\frac{1}{2}$.

NOTE.— $\sin^{-1} x$ is often written $\arcsin x$.

Exercise 48

1. (a) What is the numerically least angle whose sine is negative $\frac{1}{2}$, or expressed symbolically $\sin^{-1} (-\frac{1}{2}) = \text{what?}$ The answer is -30° .

NOTE.—The numerically least angle whose sine is x is called the principal value of $\sin^{-1} x$. We will agree that the symbol $\sin^{-1} x$ calls for the principal value only.

(b) $\cos^{-1}(-\frac{1}{2}) = \text{what?}$ In words, what is that angle whose cosine is $-\frac{1}{2}$? The answer is 120° , which is the least angle.

(c) $\tan^{-1} \sqrt{3} = \text{what?}$

(d) $\sin(\sin^{-1} \frac{1}{2}) = \text{what?}$ Beginning within the parenthesis $\sin^{-1} \frac{1}{2} = 30^\circ$ and $\sin 30^\circ = \frac{1}{2}$, therefore, $\sin(\sin^{-1} \frac{1}{2}) = \frac{1}{2}$.

(e) Show that $\cos(\sin^{-1} \frac{1}{2}) = \frac{\sqrt{3}}{2}$.

(f) Find the angles for, $\sin^{-1} 0$, $\cos^{-1} 1$, $\tan^{-1}(-1) \csc^{-1} 2$.

2. Construct $\theta = \tan^{-1} \frac{3}{4}$. Construct the right triangle ABC with $a = 3$, $b = 4$, then angle $A = \tan^{-1} \frac{3}{4}$.

3. Find $\sin(\tan^{-1} \frac{3}{4})$.

4. If $\theta = \sin^{-1} \frac{4}{5}$, find $\cos \theta$.

61. Solution of Oblique Triangles.—Two theorems are needed for the solution of oblique triangles. They are known as (1) the law of sines, (2) the law of cosines.

The Law of Sines.—When referred to a triangle ABC the law of sines is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

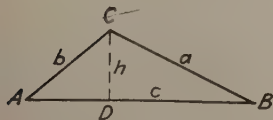


FIG. 18.

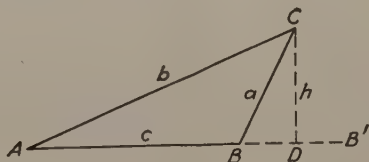


FIG. 19.

Let ABC be a triangle with a perpendicular h drawn from the vertex C to the opposite side AB (Fig. 18) or AB produced (Fig. 19). In either figure

$$\frac{h}{b} = \sin A \text{ or } h = b \sin A$$

and

$$\frac{h}{a} = \sin B \text{ or } h = a \sin B.$$

Therefore

$$h = a \sin B = b \sin A$$

and

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, by drawing a perpendicular from A to BC or BC extended, we can show that $\frac{b}{\sin B} = \frac{c}{\sin C}$.

From the last two equations we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In solving oblique triangles we apply the law of sines:

CASE I. When two angles and a side are given.

CASE II. When two sides and an angle opposite one of them are given.

CASE I.

Example.—Given $A = 46^\circ 26'$, $B = 19^\circ 42'$, and $a = 36.47$. Find C , b , and c .

Solution.— $C = 180^\circ - (A + B) = 180^\circ - 66^\circ 8'$
 $= 113^\circ 52'.$

Since angles A and B and side a are given, there is but one unknown b in the formula $b/\sin B = a/\sin A$. Solving for b and taking the logarithm of both members of the equation we have

$$\begin{aligned} \log b &= \log a + \log \sin B - \log \sin A; \\ &= \log 36.47 + \log \sin 19^\circ 42' - \log \sin 46^\circ 26' \\ &= 1.5619 + 9.5277 - 10 - (9.8601 - 10) \\ &= 1.2295. \\ b &= 16.96. \end{aligned}$$

Similarly to find side c use $c/\sin C = a/\sin A$. Then

$$\begin{aligned} \log c &= \log a + \log \sin C - \log \sin A \\ &= \log 36.47 + \log \sin 113^\circ 52' - \log \sin 46^\circ 26' \\ &= 1.5619 + 9.9612 - 10 - (9.8601 - 10) \\ &= 1.6630. \\ c &= 46.02. \end{aligned}$$

Check: Apply the law of sines so as to include the two computed sides as $b/\sin B = c/\sin C$ or $\sin B/b = \sin C/c$. By

use of the table of natural functions $\sin 19^\circ 42' = 0.3371$ and $\sin 113^\circ 52' = 0.9145$ and $\sin B/b = 0.3371/16.96 = 0.01988-$, $\sin C/c = 0.9145/46.02 = 0.01987+$.

CASE II.

Example:

Given $a = 18.25$, $b = 42.30$, and $A = 25^\circ 20'$. Notice that the given angle is acute and the side opposite it is less than the side adjacent. This is the doubtful or ambiguous case which was studied in plane geometry. A carefully constructed figure will often tell whether the triangle has two solutions, one solution, or no solution. By referring to Fig. 19 it is clear that if b is greater than a greater than h , then there are two solutions, if $a = h$, one solution and if a is less than h , no solution. Since $h = b \sin A$, the test may be stated as follows:

a is greater than $b \sin A$, two solutions.

$a = b \sin A$, one solution.

a is less than $b \sin A$, no solution.

We will apply this test to the above problem. If a is greater than $b \sin A$ then $\log a$ is greater than $\log b + \log \sin A$. $\log 18.25 = 1.2613$ and $\log 42.30 + \log \sin 25^\circ 20' = 1.6263 + 9.6313 - 10 = 1.2576$.

Therefore $\log a$ is greater than $\log b + \log \sin A$ and a is greater than $b \sin A$ and this triangle has two solutions. In other words, with the above specifications one can build two triangles.

Again returning to Fig. 19, side a may join C and B or C and B' . Since sides a and b and angle A are given there is but one unknown in the formula $\sin B'/b = \sin A/a$. (We assume that side a joins C to B' .) Solving for $\sin B'$ and taking the logarithm of both members we have

$$\begin{aligned}\log \sin B' &= \log b + \log \sin A - \log a \\ &= 1.6263 + 9.6313 - 10 - 1.2613 \\ &= 9.9963 - 10 \\ B' &= 82^\circ 30',\end{aligned}$$

and

$$B = 180^\circ - (82^\circ 30') = 97^\circ 30'.$$

When side a joins C to B then the angle $ACB = C = 180^\circ - (A + B) = 180^\circ - (122^\circ 50') = 57^\circ 10'$ and when a joins C to B' then the angle $ACB' = C' = 180^\circ - (A + B') = 180^\circ - (107^\circ 50') = 72^\circ 10'$. We now find c and c' by the sine law as follows

$$c = \frac{a \sin C}{\sin A} = \frac{18.25 \sin 57^\circ 10'}{\sin 25^\circ 20'} = 35.03$$

and

$$c' = \frac{a \sin C'}{\sin A} = \frac{18.25 \sin 72^\circ 10'}{\sin 25^\circ 30'} = 40.61.$$

The Law of Cosines.—When referred to a triangle ABC the law of cosines is:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Let ABC be a triangle with a perpendicular h drawn from the vertex C to the opposite side AB (Fig. 20) or AB produced (Fig. 21).

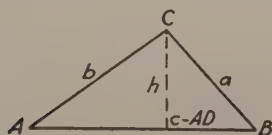


FIG. 20.

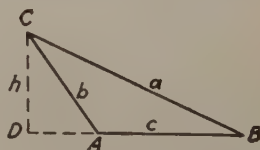


FIG. 21.

In Fig. 20.

$$\begin{aligned} a^2 &= h^2 + (c - AD)^2 \\ &= h^2 + c^2 - 2c\overline{AD} + \overline{AD}^2 \\ &= b^2 + c^2 - 2c\overline{AD} \\ &\quad \text{since } h^2 + \overline{AD}^2 = b^2 \\ &= b^2 + c^2 - 2bc \cos A \\ &\quad \text{since } AD = b \cos A \end{aligned}$$

In Fig. 21:

$$\begin{aligned} a^2 &= h^2 + (c + \overline{DA})^2 \\ &= h^2 + c^2 + 2c\overline{DA} + \overline{DA}^2 \\ &= b^2 + c^2 - 2c\overline{DA} \\ &= b^2 + c^2 + 2bc \cos (180^\circ - A) \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

Hence in either case $a^2 = b^2 + c^2 - 2bc \cos A$. The theorem is perfectly general.

Therefore

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

In solving oblique triangles we apply the cosine law:

CASE III. When two sides and the included angle are given.

CASE IV. When three sides are given.

CASE III.

Example.—Given: $a = 15.00$, $b = 12.00$, and $C = 32^\circ 20'$, find c , A , and B

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= 225.00 + 144.00 - 304.2$$

$$= 64.80$$

$$c = 8.05.$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{144.00 + 64.80 - 225.00}{193.2} = -\frac{16.20}{193.2} = -.08385.$$

$$A = 94^\circ 48'.$$

$$B = 180^\circ - (A + C) = 180^\circ - (127^\circ 8') = 52^\circ 52'.$$

We will check results by use of the law of tangents.

NOTE.—If desired check by law of sines in the form $c \sin A = a \sin C$.

Law of Tangents.—When referred to the triangle ABC the law of tangents is: $\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$.

This formula is derived as follows: By law of sines $a/b = \sin A/\sin B$. Subtracting unity from both members of the equation and simplifying we have $\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}$. By adding unity to both members and simplifying we have $\frac{a+b}{b} =$

$\frac{\sin A + \sin B}{\sin B}$. By dividing equals by equals we obtain

$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}$. By use of the results of Problem

12(c), Art. 57, $\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$. Check of above example by law of tangents.

$$a - b = 15.00 - 12.00 = 3.00.$$

$$a + b = 15.00 + 12.00 = 27.00.$$

$$A + B = 180^\circ - C = 147^\circ 40'.$$

$$\frac{1}{2}(A + B) = 73^\circ 50'.$$

It is clear that the only unknown in the formula is $\tan \frac{1}{2}(A - B)$. Therefore

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B)$$

$$\begin{aligned} \log \tan \frac{1}{2}(A - B) &= \log (a - b) + \log \tan \frac{1}{2}(A + B) - \log (a + b) \\ &= \log 3.00 + \log \tan (73^\circ 50') - \log 27.00 \\ &= 0.4771 + 0.5378 - 1.4314 \\ &= 1.0149 - 1.4314 \\ &= 11.0149 - 10 \\ &\quad - 1.4314 \\ &= 9.5835 - 10. \end{aligned}$$

$$\frac{1}{2}(A - B) = 20^\circ 58'.$$

$$A - B = 41^\circ 56'$$

But

$$A + B = 147^\circ 40'$$

Adding

$$\begin{aligned} 2A &= 189^\circ 36' \\ A &= 94^\circ 48'. \end{aligned}$$

By subtracting the equations,

$$2B = 105^\circ 44'$$

and

$$B = 52^\circ 52'.$$

As a final check

$$\begin{aligned} A + B + C &= 94^\circ 48' + 52^\circ 52' + 32^\circ 20' \\ &= 180^\circ. \end{aligned}$$

CASE IV. In working the illustrative problem in Case III after finding side c we then used the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

we assume a , b , and c to be known, to determine the angle A . This illustrates the use of the law of cosines in finding the angles of a triangle when the three sides are given.

Exercise 49

1. Work graphically and check by the law of sines: Given: $A = 38^\circ 0'$, $B = 76^\circ 30'$, $a = 12.00$; find C , b , and c .

2. Construct the following triangle to scale and apply the test for determining whether the problem has one solution or two solutions: Given:

$$b = 73.81, a = 53.65, \text{ and } A = 46^\circ 37'.$$

3. Given: $B = 56^\circ 23'$, $C = 108^\circ 50'$, $a = 246.2$ feet; find A , c , and b .

4. Given: $b = 501.3$, $A = 20^\circ 45'$, $a = 177.6$; find B and c .

5. Two objects A and B are separated by a building. In order to determine the distance between them a stake C is fixed in a position from which both A and B can be seen. The following measurements are then made:

$$AC = 146.4 \text{ feet, } BC = 120.8 \text{ feet, angle } ACB = 64^\circ 22';$$

find the distance between A and B .

Solve the following triangles by the cosine law:

6. Given: $a = 12.00$, $b = 15.00$, $C = 22^\circ 0'$.

7. Given $a = 33.10$, $c = 54.20$, $B = 30^\circ 00'$.

8. Given $a = 40.00$, $b = 22.00$, $c = 25.00$.

9. From plane geometry we know that the area of a triangle is equal to one-half the product of the base by the altitude; $A = \frac{1}{2} \text{ base} \times \text{altitude}$. Turn to Figs. 20 and 21 and show that

$$\begin{aligned} A &= \frac{1}{2}ch = \frac{1}{2}cb \sin A. \\ &= \frac{1}{2}ca \sin B. \end{aligned}$$

Verify the rule: The area of a triangle is half the product of any two sides times the sine of the included angle.

10. Use the formula in Problem 9:

- (a) Find the area of the triangle given in Problem 7.
- (b) Find the area of the triangle given in Problem 6.
- (c) Find the area of the triangle given in the first illustrative problem in Art. 61.

Supplementary Problems

1. Find graphically the unknown parts of the right triangle ABC with right angle at C and check by use of Table A, Art. 52:

Given:

- (a) $A = 60^\circ, c = 10.$
- (b) $B = 45^\circ, c = 16.$
- (c) $c = 5, a = 4.$
- (d) $c = 16, b = 10.$
- (e) $A = 75^\circ, b = 3.5.$

Required:

- | | | |
|-------|---------|---------|
| $B =$ | $, a =$ | $, b =$ |
| $A =$ | $, a =$ | $, b =$ |
| $A =$ | $, B =$ | $, b =$ |
| $A =$ | $, B =$ | $, a =$ |
| $a =$ | $, B =$ | $, c =$ |

2. Find the unknown parts of the right triangle ABC without making use of a table.

Given:

- (a) $A = 60^\circ, c = 20.$
- (b) $A = 30^\circ, a = 10.$
- (c) $a = 3, b = 3$

Required:

- | | | |
|-------|---------|---------|
| $B =$ | $, b =$ | $, a =$ |
| $B =$ | $, b =$ | $, c =$ |
| $A =$ | $, B =$ | $, c =$ |

3. From plane geometry the area of a triangle is equal to one-half the product of the base and altitude. Find the area of the right triangle ABC by making use of logarithms when

Given:

- (a) $A = 36^\circ 10', b = 14.35.$
- (b) $A = 46^\circ 32', c = 24.47.$
- (c) $b = 98.2, c = 108.7.$
- (d) $B = 45^\circ 28', b = 268.4.$

4. (a) A line AB is 264.5 feet long and when extended makes an angle of $30^\circ 0'$ with the line MN . Show that the length of the projection of AB on MN is 229.1 feet.

NOTE.—The projection of AB on MN is the distance between the perpendiculars dropped from A and B on the line MN . By use of a figure

it is clear that the projection, $p = AB \cos \theta$ where θ is the angle between MN and AB or AB produced.

- (b) A line runs northeast and is 158.0 rods long. Show that its projection on a north-and-south line is 111.7 rods.
- (c) A "dirigible" 296 feet long casts a shadow 281 feet in length when the sun is directly overhead. Show that the ship is inclined to the horizon at an angle of $18^\circ 20'$.
- (d) A force of 175 pounds acts at an angle of 60° with a line MN . What is the component of the force in the direction of this line?

NOTE.—A force of 175 pounds is represented by a line segment 175 units long in the direction of the force and the projection of this segment on MN represents the component of the force in the direction of MN .

- (e) A weight (force) of 46 pounds rests on an inclined plane AB which makes an angle of 30° with the horizontal. WH represents this force drawn to some convenient scale and WK represents the projection of WH on AB , (Fig. 22).

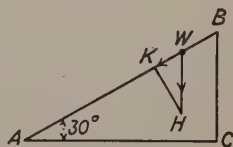


FIG. 22.

Show that $WK = 46 \cos 60^\circ = 23$. If the plane is practically frictionless, what force exerted parallel to the plane will keep the body W from slipping?

5. (a) Given $\cos A = \frac{3}{5}$. Construct all the angles less than 360° that can represent A and give the values for the sine and tangent of these angles with the correct algebraic sign of each.
- (b) Given $\tan A = -\frac{4}{3}$, find the remaining five functions of A .
6. Express each of the following as a function of the complementary angle:

(a) $\tan 60^\circ$, (b) $\cos 20^\circ$, (c) $\sin 40^\circ$, (d) $\sec 80^\circ$.

7. Express the following as functions of positive acute angles:

(a) $\sin 150^\circ$, (b) $\cos (-47^\circ)$, (c) $\tan 225^\circ$, (d) $\sec (-320^\circ)$.

8. Find without the use of tables: (a) $\sin 120^\circ$, (b) $\cos 210^\circ$, (c) $\tan 300^\circ$, (d) $\sec 135^\circ$, (e) $\cot 330^\circ$.

9. By means of the formula $\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$, find $\sin 45^\circ$ having given $\cos 90^\circ = 0$, also find $\cos 60^\circ$ having given $\sin 30^\circ = \frac{1}{2}$.

10. Make use of the tables and check the following:

(a) $\sin^2 44^\circ 50' + \cos^2 44^\circ 50' = 1$

(b) $(\cos 44^\circ 50' + \sin 44^\circ 50') (\cos 44^\circ 50' - \sin 44^\circ 50') = \cos 89^\circ 40'$.

- (c) Parts (a) and (b) are special cases of what formulas?

11. If $\csc A = 2$, find $\cos A$, $\sin A$, and $\tan A$.

Hint: On coordinate axes construct all those angles less than 360° whose cosecant is 2.

12. Prove that:

$$(a) (\sin A + \cos A)^2 = 1 + 2 \sin A \cos A = 1 + \sin 2A.$$

$$(b) (\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2.$$

13. Recall that π radians $= 180^\circ$ and with this information give the value of: (a) $\sin \pi/2$, (b) $\cos 3\pi/2$, (c) $\tan 3\pi/4$, (d) $\sin 5\pi/6$.

14. If $\sin A = \frac{1}{3}$, show that $\cos A = 0.94$, $\sin 2A = 0.63$, $\cos 2A = 0.78$.

15. By expanding show that $\sin \left(A + \frac{\pi}{2} \right) = \cos A$, $\cos \left(A + \frac{\pi}{2} \right) = -\sin A$, and $\cos \left(A - \frac{\pi}{2} \right) = \sin A$.

16. Solve the following triangles:

Given:

$$(a) a = 58.$$

$$A = 72^\circ.$$

$$b = 60.$$

$$(b) a = 22.$$

$$A = 103^\circ.$$

$$B = 20^\circ.$$

$$(c) a = 10.$$

$$A = 52^\circ.$$

$$B = 46^\circ.$$

$$(d) a = 11.$$

$$b = 20.$$

$$C = 32^\circ.$$

$$(e) b = 24.6.$$

$$c = 55.1.$$

$$A = 46^\circ.$$

$$(f) a = 12.$$

$$b = 9.$$

$$c = 6.$$

Required:

$$B =$$

$$c =$$

$$\text{area} =$$

$$C =$$

$$b =$$

$$c =$$

$$C =$$

$$b =$$

$$c =$$

$$c =$$

$$a =$$

$$\cos A =$$

$$\cos B =$$

$$\cos C =$$

$$\text{area} =$$

17. Solve the triangle:

Given:

$$A = 16^\circ 22'$$

$$C = 70^\circ 45'.$$

$$c = 254.3.$$

Required:

$$B =$$

$$a =$$

$$b =$$

$$\text{area} =$$

18. Solve the triangle:

Given:

$$a = 198.4.$$

$$b = 206.5.$$

$$C = 112^\circ 20'.$$

Required:

$$c =$$

19. Solve the triangle by use of cosine law and table of squares:

$$a = 78.2.$$

$$b = 43.6.$$

$$c = 60.4.$$

$$A =$$

$$B =$$

$$C =$$

APPENDIX

USE OF TABLES

The use of tables requires in general a knowledge of interpolation. The method of interpolation is illustrated in what follows. In making interpolations the corrected result should not contain more significant figures than are given in the table which is used. This often requires the "cutting back" of numbers. For example, in the first illustration below, the product of two differences, $0.055 \times 0.67 = 0.03685$, was cut back to 0.037 in order that the corrected result would not contain more significant figures than that part of the table from which the corrections were made. Note that the last figure retained in the correction was raised one unit. When the part cut off is equal to or greater than one-half the next higher unit, then that unit is increased by one—if less than one-half no change is made in that unit.

Table I: Powers and Roots of Numbers from 1 to 100.—Assuming that the roots of successive numbers are proportional to their corresponding numbers, we may find the root of a number that does not appear in the table, such as 83.67, by interpolation as follows: From the table

$$\sqrt{84} = 9.165.$$

$$\sqrt{83} = 9.110.$$

A difference of 1 in the numbers causes a difference of 0.055 in their roots. Therefore, a difference of 0.67 in the numbers causes a difference of 0.037 ($= 0.67 \times 0.055$) in the roots.

Therefore,

$$\sqrt{83.67} = 9.110 + 0.037 = 9.147.$$

The root of a decimal fraction such as the cube root of 0.06831 may be obtained from the table as follows:

$$\sqrt[3]{0.06831} = \sqrt[3]{\frac{68.31}{1,000}} = \frac{1}{10} \sqrt[3]{68.31}.$$

From the table

$$\sqrt[3]{69} = 4.102.$$

$$\sqrt[3]{68} = 4.082.$$

A difference of 1 in the numbers causes a difference of 0.02 in their roots. Therefore, a difference of 0.31 in the numbers causes a difference of 0.006 in the roots.

Therefore,

$$\sqrt[3]{68.31} = 4.082 + 0.006 = 4.088$$

and

$$\sqrt[3]{0.06831} = \frac{1}{10} \sqrt[3]{68.31} = 0.4088.$$

In this illustration notice that the number was first multiplied by 1,000 and the cube root of this new number was found from the table. This root when divided by 10 gave the root sought.

To find the square root of the decimal fraction, 0.06831, multiply the number by 100. Then find the root of this number (6.831) and divide this root (2.613) by 10 and obtain 0.2613 as the square root of 0.06831.

Table II: How to Find the Logarithm of a Number.—

Find log 27.4. The characteristic by rule is 1.

To obtain the mantissa from the table, look for 27 in the column headed *N*. Opposite 27 and in the column headed by 4 find the number 4378, which is the mantissa with decimal point omitted. Therefore, log 27.4 = 1.4378.

Find log 274.3. The mantissa of this logarithm is not given directly in the table. If we assume that a small change in

the number causes a proportional change in the logarithm, then we may proceed by interpolation as follows:

mantissa of log 275 is 0.4393.

mantissa of log 274 is 0.4378.

We observe that a difference of 1 in the number makes a difference of 0.0015 in the logarithm, or a difference of 0.3 in the number makes a difference of $0.0015 \times 0.3 = 0.00045$, or cutting the number back one place, our correction is 0.0005. The logarithm of $274.3 = 2.4378 + 0.0005 = 2.4383$. Time will be saved in interpolating by considering the mantissas for the moment as whole numbers. Thus, instead of writing 0.0015, write 15, and so forth.

How to Find the Number Corresponding to a Given Logarithm.—Given $\log N = 2.4383$. To find the number N . Since this problem is the converse of the preceding one, we may trace that problem back. We cannot find the mantissa 0.4383 in the table, but we find 0.4393 and 0.4378 and so forth.

Mechanical Interpolations.—In order to facilitate the computation, the tabular difference and the proportional part for the fourth figure of the natural numbers is given at the bottom of the page. The student is advised *not* to use this part of the table until he has learned to interpolate mentally with speed and accuracy. In scientific work one is called upon to use many different tables in which tabular differences and proportional parts are not given. For this reason, the student should learn to be independent of these aids.

In the above problems beginning with the tabular difference $4393 - 4378 = 15$, look at bottom of page under "Tab. Diff." for 15 which is found on the third page of the tables. Opposite 15 and in column headed 3 find 4.5. Add this correction, after cutting it back according to rule, to the mantissa 4378 and obtain 4383, the same mantissa as above with decimal point omitted.

If $\log N = 2.4383$, find the number N , proceeding as follows: Find mantissa in the table nearest less than 4383. It is 4378,

which is in column headed 4 and opposite the number 27 in the first column. Hence 0.4378 is the mantissa of the logarithm of 274.

Now

$$4383 - 4378 = 5.$$

$$4393 - 4378 = 15.$$

At bottom of page under "Tab. Diff." opposite 15, find the number nearest 5. It is 4.5 in column headed 3. Therefore, 3 is the next figure (fourth) of the number sought. The decimal point, by rule for characteristic, should be placed so that the integral part of the number will have three digits. Therefore, the number is 274.3.

Use of Tables of Trigonometric Functions

In Table III. are given the natural and logarithmic functions of angles for every 10 minutes in the quadrant. The angles less than 45° are found in the left-hand column. The functions are given in same line with the angle. Angles from 45° to 90° are found in the right-hand column. It should be noted that when angles are read on left, function names are to be read at top of page and when angles are read in right column, function names are to be read at bottom of page.

To Find the Logarithmic Sine of $41^\circ 20'$.—Since this angle is less than 45° read the angle in left column. On the line of $41^\circ 20'$ in column headed log sine find

$$\log \sin 41^\circ 20' = 9.8198.$$

The table is so constructed that the logarithmic functions are 10 larger than their actual values. This will be understood if the logarithm of the natural sine of $41^\circ 20'$ is found and its logarithm found in Table II. The adding of 10 to the logarithmic functions is a matter of facility in calculating with them.

To Find an Angle Corresponding to a Given Function.—We proceed in a manner similar to finding a number when its logarithm is given.

I. POWERS AND ROOTS OF NUMBERS FROM 1 TO 100¹

No.	Square.	Cube.	Square root.	Cube root.
1	1	1	1.000	1.000
2	4	8	1.414	1.260
3	9	27	1.732	1.442
4	16	64	2.000	1.587
5	25	125	2.236	1.710
6	36	216	2.450	1.817
7	49	343	2.646	1.913
8	64	512	2.828	2.000
9	81	729	3.000	2.080
10	100	1000	3.162	2.154
11	121	1331	3.317	2.224
12	144	1728	3.464	2.289
13	169	2197	3.606	2.351
14	196	2744	3.742	2.410
15	225	3375	3.873	2.466
16	256	4096	4.000	2.520
17	289	4913	4.123	2.571
18	324	5832	4.243	2.621
19	361	6859	4.359	2.668
20	400	8000	4.472	2.714
21	441	9261	4.583	2.759
22	484	10648	4.690	2.802
23	529	12167	4.796	2.844
24	576	13824	4.899	2.885
25	625	15625	5.000	2.924
26	676	17576	5.099	2.963
27	729	19683	5.196	3.000
28	784	21952	5.292	3.037
29	841	24389	5.385	3.072
30	900	27000	5.477	3.107
31	961	29791	5.568	3.141
32	1024	32768	5.657	3.175
33	1089	35937	5.745	3.208
34	1156	39304	5.831	3.240
35	1225	42875	5.916	3.271
36	1296	46656	6.000	3.302
37	1369	50653	6.083	3.332
38	1444	54872	6.164	3.362
39	1521	59319	6.245	3.391
40	1600	64000	6.325	3.420
41	1681	68921	6.403	3.448
42	1764	74088	6.481	3.476
43	1849	79507	6.557	3.503
44	1936	85184	6.633	3.530
45	2025	91125	6.708	3.557
46	2116	97336	6.782	3.583
47	2209	103823	6.856	3.609
48	2304	110592	6.928	3.634
49	2401	117649	7.000	3.659
50	2500	125000	7.071	3.684

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I. POWERS AND ROOTS OF NUMBERS FROM 1 TO 100. (*Cont'd*)

No.	Square.	Cube.	Square root.	Cube root.
51	2601	132651	7.141	3.708
52	2704	140608	7.211	3.733
53	2809	148877	7.280	3.756
54	2916	157464	7.349	3.780
55	3025	166375	7.416	3.803
56	3136	175616	7.483	3.826
57	3249	185193	7.550	3.849
58	3364	195112	7.616	3.871
59	3481	205379	7.681	3.893
60	3600	216000	7.746	3.915
61	3721	226981	7.810	3.937
62	3844	238328	7.874	3.958
63	3969	250047	7.937	3.979
64	4096	262144	8.000	4.000
65	4225	274625	8.062	4.021
66	4356	287496	8.124	4.041
67	4489	300763	8.185	4.062
68	4624	314432	8.246	4.082
69	4761	328509	8.306	4.102
70	4900	343000	8.367	4.121
71	5041	357911	8.426	4.141
72	5184	373248	8.485	4.160
73	5329	389017	8.544	4.179
74	5476	405224	8.602	4.198
75	5625	421875	8.660	4.217
76	5776	438976	8.718	4.236
77	5929	456533	8.775	4.254
78	6084	474552	8.832	4.273
79	6241	493039	8.888	4.291
80	6400	512000	8.944	4.309
81	6561	531441	9.000	4.327
82	6724	551368	9.055	4.345
83	6889	571787	9.110	4.362
84	7056	592704	9.165	4.380
85	7225	614125	9.220	4.397
86	7396	636056	9.274	4.414
87	7569	658503	9.327	4.431
88	7744	681472	9.381	4.448
89	7921	704969	9.434	4.465
90	8100	729000	9.487	4.481
91	8281	753571	9.539	4.498
92	8464	778688	9.592	4.514
93	8649	804357	9.644	4.531
94	8836	830584	9.695	4.547
95	9025	857375	9.747	4.563
96	9216	884736	9.798	4.579
97	9409	912673	9.849	4.595
98	9604	941192	9.900	4.610
99	9801	970299	9.950	4.626
100	10000	1000000	10.000	4.642

II. LOGARITHMS¹

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010

Tab. diff.	Extra digit.								
	1	2	3	4	5	6	7	8	9
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1
38	3.8	7.6	11.4	15.2	19.0	22.8	26.6	30.4	34.2
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3
36	3.6	7.2	10.8	14.4	18.0	21.6	25.2	28.8	32.4
35	3.5	7.0	10.5	14.0	17.5	21.0	24.5	28.0	31.5
34	3.4	6.8	10.2	13.6	17.0	20.4	23.8	27.2	30.6
33	3.3	6.6	9.9	13.2	16.5	19.8	23.1	26.4	29.7
32	3.2	6.4	9.6	12.8	16.0	19.2	22.4	25.6	28.8
31	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9
30	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0

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II. LOGARITHMS (Continued)

No.	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445

Extra digit.

Tab. diff.	1	2	3	4	5	6	7	8	9
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1
28	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
27	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
26	2.6	5.2	7.8	10.4	13.0	15.6	18.2	20.8	23.4
25	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5
24	2.4	4.8	7.2	9.6	12.0	14.4	16.8	19.2	21.6
23	2.3	4.6	6.9	9.2	11.5	13.8	16.1	18.4	20.7
22	2.2	4.4	6.6	8.8	11.0	13.2	15.4	17.6	19.8
21	2.1	4.2	6.3	8.4	10.5	12.6	14.7	16.8	18.9
20	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1
18	1.8	3.6	5.4	7.2	9.0	10.8	12.6	14.4	16.2
17	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4

II. LOGARITHMS (Continued)

No.	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Tab. Diff.	Extra digit.								
	1	2	3	4	5	6	7	8	9
15	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5
14	1.4	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6
13	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
10	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
8	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
6	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
4	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6

III. NATURAL AND LOGARITHMIC FUNCTIONS¹

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0°	0.0000	—∞	1.0000	0.0000	0.0000	—∞	∞		90°
10'	.0029	7.4637	1.0000	.0000	.0029	7.4637	343.8	12.5363	50'
20'	.0058	.7648	1.0000	.0000	.0058	.7648	171.9	.2352	40'
30'	.0087	.9408	1.0000	.0000	.0087	.9409	114.6	.0591	30'
40'	.0116	8.0658	0.9999	.0000	.0116	8.0658	85.94	11.9342	20'
50'	.0145	.1627	.9999	.0000	.0145	.1627	68.75	.8373	10'
1°	.0175	.2419	.9998	9.9999	.0175	.2419	57.29	.7581	89°
10'	.0204	.3088	.9998	.9999	.0204	.3089	49.10	.6911	50'
20'	.0233	.3668	.9997	.9999	.0233	.3669	42.96	.6331	40'
30'	.0262	.4179	.9997	.9999	.0262	.4181	38.19	.5819	30'
40'	.0291	.4637	.9996	.9998	.0291	.4638	34.37	.5362	20'
50'	.0320	.5050	.9995	.9998	.0320	.5053	31.24	.4947	10'
2°	.0349	.5428	.9994	.9997	.0349	.5431	28.64	.4569	88°
10'	.0378	.5776	.9993	.9997	.0378	.5779	26.43	.4221	50'
20'	.0407	.6097	.9992	.9996	.0407	.6101	24.51	.3899	40'
30'	.0436	.6397	.9990	.9996	.0437	.6401	22.90	.3599	30'
40'	.0465	.6677	.9989	.9995	.0466	.6682	21.47	.3318	20'
50'	.0494	.6940	.9988	.9995	.0495	.6954	20.21	.3055	10'
3°	.0523	.7188	.9986	.9994	.0524	.7194	19.08	.2806	87°
10'	.0552	.7423	.9985	.9993	.0553	.7429	18.08	.2571	50'
20'	.0581	.7645	.9983	.9993	.0582	.7652	17.17	.2348	40'
30'	.0610	.7857	.9981	.9992	.0612	.7865	16.35	.2135	30'
40'	.0640	.8059	.9980	.9991	.0641	.8067	15.61	.1933	20'
50'	.0669	.8251	.9978	.9990	.0670	.8261	14.92	.1739	10'
4°	.0698	.8436	.9976	.9989	.0699	.8446	14.30	.1554	86°
10'	.0727	.8613	.9974	.9989	.0729	.8624	13.73	.1376	50'
20'	.0756	.8783	.9971	.9988	.0758	.8795	13.20	.1205	40'
30'	.0785	.8946	.9969	.9987	.0787	.8960	12.71	.1040	30'
40'	.0814	.9104	.9967	.9986	.0816	.9118	12.25	.0882	20'
50'	.0843	.9256	.9964	.9985	.0846	.9272	11.83	.0728	10'
5°	.0872	.9403	.9962	.9983	.0875	.9420	11.43	.0580	85°
10'	.0901	.9545	.9959	.9982	.0904	.9563	11.06	.0437	50'
20'	.0929	.9682	.9957	.9981	.0934	.9701	10.71	.0299	40'
30'	.0958	.9816	.9954	.9980	.0963	.9836	10.39	.0164	30'
40'	.0987	.9945	.9951	.9979	.0992	.9966	10.08	.0034	20'
50'	.1016	9.0070	.9948	.9977	.1022	9.0093	9.788	10.9907	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

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III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
6°	0.1045	9.0192	0.9945	9.9976	0.1051	9.0126	9.514	10.9784	84°
10'	.1074	.0311	.9942	.9975	.1080	.0336	9.255	.9664	50'
20'	.1103	.0426	.9939	.9973	.1110	.0453	9.010	.9547	40'
30'	.1132	.0539	.9936	.9972	.1139	.0567	8.777	.9433	30'
40'	.1161	.0648	.9932	.9971	.1169	.0678	8.556	.9322	20'
50'	.1190	.0755	.9929	.9969	.1198	.0786	8.345	.9214	10'
7°	.1219	.0859	.9925	.9968	.1228	.0891	8.144	.9109	83°
10'	.1248	.0961	.9922	.9966	.1257	.0995	7.953	.9005	50'
20'	.1276	.1060	.9918	.9964	.1287	.1096	7.770	.8904	40'
30'	.1305	.1157	.9914	.9963	.1317	.1194	7.596	.8806	30'
40'	.1334	.1252	.9911	.9961	.1346	.1291	7.429	.8709	20'
50'	.1363	.1345	.9907	.9959	.1376	.1385	7.269	.8615	10'
8°	.1392	.1436	.9903	.9958	.1405	.1478	7.115	.8522	82°
10'	.1421	.1525	.9899	.9956	.1435	.1569	6.968	.8431	50'
20'	.1449	.1612	.9894	.9954	.1465	.1658	6.827	.8342	40'
30'	.1478	.1697	.9890	.9952	.1495	.1745	6.691	.8255	30'
40'	.1507	.1781	.9886	.9950	.1524	.1831	6.561	.8169	20'
50'	.1536	.1863	.9881	.9948	.1554	.1915	6.435	.8085	10'
9°	.1564	.1943	.9877	.9946	.1584	.1997	6.314	.8003	81°
10'	.1593	.2022	.9872	.9944	.1614	.2078	6.197	.7922	50'
20'	.1622	.2100	.9868	.9942	.1644	.2158	6.084	.7842	40'
30'	.1650	.2176	.9863	.9940	.1673	.2236	5.976	.7764	30'
40'	.1679	.2251	.9858	.9938	.1703	.2313	5.871	.7687	20'
50'	.1708	.2324	.9853	.9936	.1733	.2389	5.769	.7611	10'
10°	.1736	.2397	.9848	.9934	.1763	.2463	5.671	.7537	80°
10'	.1765	.2468	.9843	.9931	.1793	.2536	5.576	.7464	50'
20'	.1794	.2538	.9838	.9929	.1823	.2609	5.485	.7391	40'
30'	.1822	.2606	.9833	.9927	.1853	.2680	5.396	.7320	30'
40'	.1851	.2674	.9827	.9924	.1883	.2750	5.309	.7250	20'
50'	.1880	.2740	.9822	.9922	.1914	.2819	5.226	.7181	10'
11°	.1908	.2806	.9816	.9919	.1944	.2887	5.145	.7113	79°
10'	.1937	.2870	.9811	.9917	.1974	.2953	5.066	.7047	50'
20'	.1965	.2934	.9805	.9914	.2004	.3020	4.989	.6980	40'
30'	.1994	.2997	.9799	.9912	.2035	.3085	4.915	.6915	30'
40'	.2022	.3058	.9793	.9909	.2065	.3149	4.843	.6851	20'
50'	.2051	.3119	.9787	.9907	.2095	.3212	4.773	.6788	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
12°	0.2079	9.3179	0.9781	9.9904	0.2126	9.3275	4.705	10.6725	78°
10'	.2108	.3238	.9775	.9901	.2156	.3336	4.638	.6664	50'
20'	.2136	.3296	.9769	.9899	.2186	.3397	4.574	.6603	40'
30'	.2164	.3353	.9763	.9896	.2217	.3458	4.511	.6542	30'
40'	.2193	.3410	.9757	.9893	.2247	.3517	4.449	.6483	20'
50'	.2221	.3466	.9750	.9890	.2278	.3576	4.390	.6424	10'
13°	.2250	.3521	.9744	.9887	.2309	.3634	4.332	.6366	77°
10'	.2278	.3575	.9737	.9884	.2339	.3691	4.275	.6309	50'
20'	.2306	.3629	.9730	.9881	.2370	.3748	4.219	.6252	40'
30'	.2334	.3682	.9724	.9878	.2401	.3804	4.165	.6196	30'
40'	.2363	.3734	.9717	.9875	.2432	.3859	4.113	.6141	20'
50'	.2391	.3786	.9710	.9872	.2462	.3914	4.061	.6086	10'
14°	.2419	.3837	.9703	.9869	.2493	.3968	4.011	.6032	76°
10'	.2447	.3887	.9696	.9866	.2524	.4021	3.962	.5979	50'
20'	.2476	.3937	.9689	.9863	.2555	.4074	3.914	.5926	40'
30'	.2504	.3986	.9681	.9859	.2586	.4127	3.867	.5873	30'
40'	.2532	.4035	.9674	.9856	.2617	.4178	3.821	.5822	20'
50'	.2560	.4083	.9667	.9853	.2648	.4230	3.776	.5770	10'
15°	.2588	.4130	.9659	.9849	.2679	.4281	3.732	.5719	75°
10'	.2616	.4177	.9652	.9846	.2711	.4331	3.689	.5669	50'
20'	.2644	.4223	.9644	.9843	.2742	.4381	3.647	.5619	40'
30'	.2672	.4269	.9636	.9839	.2773	.4430	3.606	.5570	30'
40'	.2700	.4314	.9628	.9836	.2805	.4479	3.566	.5521	20'
50'	.2728	.4359	.9621	.9832	.2836	.4527	3.526	.5473	10'
16°	.2756	.4403	.9613	.9828	.2867	.4575	3.487	.5425	74°
10'	.2784	.4447	.9605	.9825	.2899	.4622	3.450	.5378	50'
20'	.2812	.4491	.9596	.9821	.2931	.4669	3.412	.5331	40'
30'	.2840	.4533	.9588	.9817	.2962	.4716	3.376	.5284	30'
40'	.2868	.4576	.9580	.9814	.2994	.4762	3.340	.5238	20'
50'	.2896	.4618	.9572	.9810	.3026	.4808	3.305	.5192	10'
17°	.2924	.4659	.9563	.9806	.3057	.4853	3.271	.5147	73°
10'	.2952	.4700	.9555	.9802	.3089	.4898	3.237	.5102	50'
20'	.2979	.4741	.9546	.9798	.3121	.4943	3.204	.5057	40'
30'	.3007	.4781	.9537	.9794	.3153	.4987	3.172	.5013	30'
40'	.3035	.4821	.9528	.9790	.3185	.5031	3.140	.4969	20'
50'	.3062	.4861	.9520	.9786	.3217	.5075	3.108	.4925	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18°	0.3090	9.4900	0.9511	9.9782	0.3249	9.5118	3.078	10.4882	72°
10'	.3118	.4939	.9502	.9778	.3281	.5161	3.048	.4839	50'
20'	.3145	.4977	.9492	.9774	.3314	.5203	3.018	.4797	40'
30'	.3173	.5015	.9483	.9770	.3346	.5245	2.989	.4755	30'
40'	.3201	.5052	.9474	.9765	.3378	.5287	2.960	.4713	20'
50'	.3228	.5090	.9465	.9761	.3411	.5329	2.932	.4671	10'
19°	.3256	.5126	.9455	.9757	.3443	.5370	2.904	.4630	71°
10'	.3283	.5163	.9446	.9752	.3476	.5411	2.877	.4589	50'
20'	.3311	.5199	.9436	.9748	.3508	.5451	2.850	.4549	40'
30'	.3338	.5235	.9426	.9743	.3541	.5491	2.824	.4509	30'
40'	.3365	.5270	.9417	.9739	.3574	.5531	2.798	.4469	20'
50'	.3393	.5306	.9407	.9734	.3607	.5571	2.773	.4429	10'
20°	.3420	.5341	.9397	.9730	.3640	.5611	2.748	.4389	70°
10'	.3448	.5375	.9387	.9725	.3673	.5650	2.723	.4350	50'
20'	.3475	.5409	.9377	.9721	.3706	.5689	2.699	.4311	40'
30'	.3502	.5443	.9367	.9716	.3739	.5727	2.675	.4273	30'
40'	.3529	.5477	.9356	.9711	.3772	.5766	2.651	.4234	20'
50'	.3557	.5510	.9346	.9706	.3805	.5804	2.628	.4196	10'
21°	.3584	.5543	.9336	.9702	.3839	.5842	2.605	.4158	69°
10'	.3611	.5576	.9325	.9697	.3872	.5879	2.583	.4121	50'
20'	.3638	.5609	.9315	.9692	.3906	.5917	2.561	.4083	40'
30'	.3665	.5641	.9304	.9687	.3939	.5954	2.539	.4046	30'
40'	.3692	.5673	.9293	.9682	.3973	.5991	2.517	.4009	20'
50'	.3719	.5704	.9283	.9677	.4006	.6028	2.496	.3972	10'
22°	.3746	.5736	.9272	.9672	.4040	.6064	2.475	.3936	68°
10'	.3773	.5767	.9261	.9667	.4074	.6100	2.455	.3900	50'
20'	.3800	.5798	.9250	.9661	.4108	.6136	2.434	.3864	40'
30'	.3827	.5828	.9239	.9656	.4142	.6172	2.414	.3828	30'
40'	.3854	.5859	.9228	.9651	.4176	.6208	2.395	.3792	20'
50'	.3881	.5889	.9216	.9646	.4210	.6243	2.375	.3757	10'
23°	.3907	.5919	.9205	.9640	.4245	.6279	2.356	.3721	67°
10'	.3934	.5948	.9194	.9635	.4279	.6314	2.337	.3686	50'
20'	.3961	.5978	.9182	.9629	.4314	.6348	2.318	.3652	40'
30'	.3987	.6007	.9171	.9624	.4348	.6383	2.300	.3617	30'
40'	.4014	.6036	.9159	.9618	.4383	.6417	2.282	.3583	20'
50'	.4041	.6065	.9147	.9613	.4417	.6452	2.264	.3548	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of function at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (Continued)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
24°	0.4067	9.6093	0.9135	9.9607	0.4452	9.6486	2.246	10.3514	66°
10'	.4094	.6121	.9124	.9602	.4487	.6520	2.229	.3480	50'
20'	.4120	.6149	.9112	.9596	.4522	.6553	2.211	.3447	40'
30'	.4147	.6177	.9100	.9590	.4557	.6587	2.194	.3413	30'
40'	.4173	.6205	.9088	.9584	.4592	.6620	2.178	.3380	20'
50'	.4200	.6232	.9075	.9579	.4628	.6654	2.161	.3346	10'
25°	.4226	.6259	.9063	.9573	.4663	.6687	2.145	.3313	65°
10'	.4253	.6286	.9051	.9567	.4699	.6720	2.128	.3280	50'
20'	.4279	.6313	.9038	.9561	.4734	.6752	2.112	.3248	40'
30'	.4305	.6340	.9026	.9555	.4770	.6785	2.097	.3215	30'
40'	.4331	.6366	.9013	.9549	.4806	.6817	2.081	.3183	20'
50'	.4358	.6392	.9001	.9543	.4841	.6850	2.066	.3150	10'
26°	.4384	.6418	.8988	.9537	.4877	.6882	2.050	.3118	64°
10'	.4410	.6444	.8975	.9530	.4913	.6914	2.035	.3086	50'
20'	.4436	.6470	.8962	.9524	.4950	.6946	2.020	.3054	40'
30'	.4462	.6495	.8949	.9518	.4986	.6977	2.006	.3023	30'
40'	.4488	.6521	.8936	.9512	.5022	.7009	1.991	.2991	20'
50'	.4514	.6546	.8923	.9505	.5059	.7040	1.977	.2960	10'
27°	.4540	.6570	.8910	.9499	.5095	.7072	1.963	.2928	63°
10'	.4566	.6595	.8897	.9492	.5132	.7103	1.949	.2897	50'
20'	.4592	.6620	.8884	.9486	.5169	.7134	1.935	.2866	40'
30'	.4617	.6644	.8870	.9479	.5206	.7165	1.921	.2835	30'
40'	.4643	.6668	.8857	.9473	.5243	.7196	1.907	.2804	20'
50'	.4669	.6692	.8843	.9466	.5280	.7226	1.894	.2774	10'
28°	.4695	.6716	.8829	.9459	.5317	.7257	1.881	.2743	62°
10'	.4720	.6740	.8816	.9453	.5354	.7287	1.868	.2713	50'
20'	.4746	.6763	.8802	.9446	.5392	.7317	1.855	.2683	40'
30'	.4772	.6787	.8788	.9439	.5430	.7348	1.842	.2652	30'
40'	.4797	.6810	.8774	.9432	.5467	.7378	1.829	.2622	20'
50'	.4823	.6833	.8760	.9425	.5505	.7408	1.817	.2592	10'
29°	.4848	.6856	.8746	.9418	.5543	.7438	1.804	.2562	61°
10'	.4874	.6878	.8732	.9411	.5581	.7467	1.792	.2533	50'
20'	.4899	.6901	.8718	.9404	.5619	.7497	1.780	.2503	40'
30'	.4924	.6923	.8704	.9397	.5658	.7526	1.768	.2474	30'
40'	.4950	.6946	.8689	.9390	.5696	.7556	1.756	.2444	20'
50'	.4975	.6968	.8675	.9383	.5735	.7585	1.744	.2415	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
30°	0.5000	9.6990	0.8660	9.9375	0.7774	9.7614	1.732	10.2386	60°
10'	.5025	.7012	.8646	.9368	.5812	.7644	1.721	.2356	50'
20'	.5050	.7033	.8631	.9361	.5851	.7673	1.709	.2327	40'
30'	.5075	.7055	.8616	.9353	.5890	.7701	1.698	.2299	30'
40'	.5100	.7076	.8601	.9346	.5930	.7730	1.686	.2270	20'
50'	.5125	.7097	.8587	.9338	.5969	.7759	1.675	.2241	10'
31°	.5150	.7118	.8572	.9331	.6009	.7788	1.664	.2212	59°
10'	.5175	.7139	.8557	.9323	.6048	.7816	1.653	.2184	50'
20'	.5200	.7160	.8542	.9315	.6088	.7845	1.643	.2155	40'
30'	.5225	.7181	.8526	.9308	.6128	.7873	1.632	.2127	30'
40'	.5250	.7201	.8511	.9300	.6168	.7902	1.621	.2098	20'
50'	.5275	.7222	.8496	.9292	.6208	.7930	1.611	.2070	10'
32°	.5299	.7242	.8480	.9284	.6249	.7958	1.600	.2042	58°
10'	.5324	.7262	.8465	.9276	.6289	.7986	1.590	.2014	50'
20'	.5348	.7282	.8450	.9268	.6330	.8014	1.580	.1986	40'
30'	.5373	.7302	.8434	.9260	.6371	.8042	1.570	.1958	30'
40'	.5398	.7322	.8418	.9252	.6412	.8070	1.560	.1930	20'
50'	.5422	.7342	.8403	.9244	.6453	.8097	1.550	.1903	10'
33°	.5446	.7361	.8387	.9236	.6494	.8125	1.540	.1875	57°
10'	.5471	.7380	.8371	.9228	.6536	.8153	1.530	.1847	50'
20'	.5495	.7400	.8355	.9219	.6577	.8180	1.520	.1820	40'
30'	.5519	.7419	.8339	.9211	.6619	.8208	1.511	.1792	30'
40'	.5544	.7438	.8323	.9203	.6661	.8235	1.501	.1765	20'
50'	.5568	.7457	.8307	.9194	.6703	.8263	1.492	.1737	10'
34°	.5592	.7476	.8290	.9186	.6745	.8290	1.483	.1710	56°
10'	.5616	.7494	.8274	.9177	.6787	.8317	1.473	.1683	50'
20'	.5640	.7513	.8258	.9169	.6830	.8344	1.464	.1656	40'
30'	.5664	.7531	.8241	.9160	.6873	.8371	1.455	.1629	30'
40'	.5688	.7550	.8225	.9151	.6916	.8398	1.446	.1602	20'
50'	.5712	.7568	.8208	.9142	.6959	.8425	1.437	.1575	10'
35°	.5736	.7586	.8192	.9134	.7002	.8452	1.428	.1548	55°
10'	.5760	.7604	.8175	.9125	.7046	.8479	1.419	.1521	50'
20'	.5783	.7622	.8158	.9116	.7089	.8506	1.411	.1494	40'
30'	.5807	.7640	.8141	.9107	.7133	.8533	1.402	.1467	30'
40'	.5831	.7657	.8124	.9098	.7177	.8559	1.393	.1441	20'
50'	.5854	.7675	.8107	.9089	.7221	.8586	1.385	.1414	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36°	0.5878	9.7692	0.8090	9.9080	0.7265	9.8613	1.376	10.1387	54°
10'	.5901	.7710	.8073	.9070	.7310	.8639	1.368	.1361	50'
20'	.5925	.7727	.8056	.9061	.7355	.8666	1.360	.1334	40'
30'	.5948	.7744	.8039	.9052	.7400	.8692	1.351	.1308	30'
40'	.5972	.7761	.8021	.9042	.7445	.8718	1.343	.1282	20'
50'	.5995	.7778	.8004	.9033	.7490	.8745	1.335	.1255	10'
37°	.6018	.7795	.7986	.9023	.7536	.8771	1.327	.1229	53°
10'	.6041	.7811	.7969	.9014	.7581	.8797	1.319	.1203	50'
20'	.6065	.7828	.7951	.9004	.7627	.8824	1.311	.1176	40'
30'	.6088	.7844	.7934	.8995	.7673	.8850	1.303	.1150	30'
40'	.6111	.7861	.7916	.8985	.7720	.8876	1.295	.1124	20'
50'	.6134	.7877	.7898	.8975	.7766	.8902	1.288	.1098	10'
38°	.6157	.7893	.7880	.8965	.7813	.8928	1.280	.1072	52°
10'	.6180	.7910	.7862	.8955	.7860	.8954	1.272	.1046	50'
20'	.6202	.7926	.7844	.8945	.7907	.8980	1.265	.1020	40'
30'	.6225	.7941	.7826	.8935	.7954	.9006	1.257	.0994	30'
40'	.6248	.7957	.7808	.8925	.8002	.9032	1.250	.0968	20'
50'	.6271	.7973	.7790	.8915	.8050	.9058	1.242	.0942	10'
39°	.6293	.7989	.7771	.8905	.8098	.9084	1.235	.0916	51°
10'	.6316	.8004	.7753	.8895	.8146	.9110	1.228	.0890	50'
20'	.6338	.8020	.7735	.8884	.8195	.9135	1.220	.0865	40'
30'	.6361	.8035	.7716	.8874	.8243	.9161	1.213	.0839	30'
40'	.6383	.8050	.7698	.8864	.8292	.9187	1.206	.0813	20'
50'	.6406	.8066	.7679	.8853	.8342	.9212	1.199	.0788	10'
40°	.6428	.8081	.7660	.8843	.8391	.9238	1.192	.0762	50°
10'	.6450	.8096	.7642	.8832	.8441	.9264	1.185	.0736	50'
20'	.6472	.8111	.7623	.8821	.8491	.9289	1.178	.0711	40'
30'	.6494	.8125	.7604	.8810	.8541	.9315	1.171	.0685	30'
40'	.6517	.8140	.7585	.8800	.8591	.9341	1.164	.0659	20'
50'	.6539	.8155	.7566	.8789	.8642	.9366	1.157	.0634	10'
41°	.6561	.8169	.7547	.8778	.8693	.9392	1.150	.0608	49°
10'	.6583	.8184	.7528	.8767	.8744	.9417	1.144	.0583	50'
20'	.6604	.8198	.7509	.8756	.8796	.9443	1.137	.0557	40'
30'	.6626	.8213	.7490	.8745	.8847	.9468	1.130	.0532	30'
40'	.6648	.8227	.7470	.8733	.8899	.9494	1.124	.0506	20'
50'	.6670	.8241	.7451	.8722	.8952	.9519	1.117	.0481	10'
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (Continued)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
42°	0.6691	9.8255	0.7431	9.8711	0.9004	9.9544	1.111	10.0456	48°
10'	.6713	.8269	.7412	.8699	.9057	.9570	1.104	.0430	50'
20'	.6734	.8283	.7392	.8688	.9110	.9595	1.098	.0405	40'
30'	.6756	.8297	.7373	.8676	.9163	.9621	1.091	.0379	30'
40'	.6777	.8311	.7353	.8665	.9217	.9646	1.085	.0354	20'
50'	.6799	.8324	.7333	.8653	.9271	.9671	1.079	.0329	10'
43°	.6820	.8338	.7314	.8641	.9325	.9697	1.072	.0303	47°
10'	.6841	.8351	.7294	.8629	.9380	.9722	1.066	.0278	50'
20'	.6862	.8365	.7274	.8618	.9435	.9747	1.060	.0253	40'
30'	.6884	.8378	.7254	.8606	.9490	.9772	1.054	.0228	30'
40'	.6905	.8391	.7234	.8594	.9545	.9798	1.048	.0202	20'
50'	.6926	.8405	.7214	.8582	.9601	.9823	1.042	.0177	10'
44°	.6947	.8418	.7193	.8569	.9657	.9848	1.036	.0152	46°
10'	.6967	.8431	.7173	.8557	.9713	.9874	1.030	.0126	50'
20'	.6988	.8444	.7153	.8545	.9770	.9899	1.024	.0101	40'
30'	.7009	.8457	.7133	.8532	.9827	.9924	1.018	.0076	30'
40'	.7030	.8469	.7112	.8520	.9884	.9949	1.012	.0051	20'
50'	.7050	.8482	.7092	.8507	.9942	.9975	1.006	.0025	10'
45°	.7071	.8495	.7071	.8495	1.0000	.0000	1.000	.0000	45°
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

IV. CONVERSION TABLE

Radians to degrees		Degrees to radians	
0.1	5° 44'	0° 10'	0.00291
0.2	11 28	0 20	0.00582
0.3	17 11	0 30	0.00873
0.4	22 55	1	0.01745
0.5	28 39	2	0.03491
0.6	34 23	3	0.05236
0.7	40 06	4	0.06981
0.8	45 50	5	0.08727
0.9	51 34	10	0.17453
1.0	57 18	20	0.34907
2.0	114 35	30	0.52360
3.0	171 53	40	0.69813
4.0	229 11	50	0.87266
5.0	286 29	57 18	1.00000
6.0	343 46	60	1.04720
7.0	401 04	90	1.57080
8.0	458 22		
9.0	515 40		

FORMULAS FOR REFERENCE

Explanatory Representations

A = area.	$\pi = 3.1416$ or approxi-	r = radius.
d = diameter or diagonal.	mately $3\frac{1}{2}$	L = lateral area.
h = height or altitude.	b, b' = bases.	s = slant height.
a, b, c = sides of $\triangle ABC$.	p, p' = perimeters	C = circumference.
S = semiperimeter of a	e = lateral edge	
triangle = $\frac{1}{2}(a + b + c)$.		

Line Values

Circle, $C = \pi d$ or $2\pi r$.Radius of a circle, $r = C/2\pi$.Right triangle, $a^2 + b^2 = c^2$.Equilateral triangle, $h = \frac{b}{2}\sqrt{3}$.Diagonal of square, $d = b\sqrt{2}$.Side of square, $b = \sqrt{A}$.

Areas of Plane Figures

Rectangle, bh .Square, b^2 .Parallelogram, bh Trapezoid, $\frac{h(b + b')}{2}$.Triangle, $bh/2$ or $\sqrt{S(S-a)(S-b)(S-c)}$. Circle, $\frac{rC}{2}$ or πr^2 .Equilateral triangle, $\frac{b^2}{4}\sqrt{3}$.

Areas of Solids

Prism, $L = ep$ Regular pyramid, $L = \frac{sp}{2}$.Frustum of regular pyramid, $L = \frac{(p + p')s}{2}$.Cylinder of revolution, $L = hC = 2\pi rh$.Cone of revolution, $L = \frac{sC}{2} = \pi rs$.Frustum of cone of revolution, $L = \frac{s(C + C')}{2}$.Sphere, $A = 4\pi r^2$.

Volumes

Rectangular parallelopiped, bh . Pyramid or cone, $\frac{bh}{3}$.Prism or cylinder, bh .Cone of revolution, $\frac{\pi r^2 h}{3}$.Right-circular cylinder, $\pi r^2 h$.Sphere, $\frac{4\pi r^3}{3} = \frac{\pi d^3}{6}$.Frustum of pyramid or cone, $\frac{h(b + b' + \sqrt{bb'})}{3}$.Frustum of cone of revolution, $\frac{\pi h(r^2 + r'^2 + rr')}{3}$.

Square Measure

144 square inches = 1 square foot.

9 square feet = 1 square yard.

30 $\frac{1}{4}$ square yards or 272 $\frac{1}{4}$ square feet =
1 square rod.160 square rods or 43,560 square feet =
1 acre.

640 acres = 1 square mile.

Units of Measure

1 meter = 10 decimeters.

= 100 centimeters.

= 1,000 millimeters.

1 meter = 39.37 inches.

 $\frac{C}{100} = \frac{(F - 32)}{180}$ where,

C. = degrees Centigrade.

and F. = degrees Fahrenheit.

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ANSWERS

Exercise 2. Page 7

3. $(2x + 3)(5x + 4); (5x + 4)(2x - 3); (8x + 15)(6x + 7).$
4. $x(x + 2)(x + 1).$
5. $(a + b + c)(a + b - c)(a - b + c)(b + c - a).$
6. $(g - k^2)(t - x).$
7. $(x + 2y)(x^2 - 2xy + 4y^2).$
8. $(3x + 2y)(3x + 2y).$
9. $(a - b)(a - 2b)(a^2 - 2ab + 4b^2).$
10. $-4n^2m^2.$
11. $(3y - 2b)(3y - 2b)(9y^2 + 6by + 4b^2)(9y^2 + 6by + 4b^2).$
12. $v(t - v).$
13. $v(v - t)(v + t).$
14. $(t - v)(t^2 + vt + v^2).$
15. $(v - t)(v - t).$
16. $3(u - 3v)(u + v).$
17. $(x - 3y)(x + 3y)(x^2 + 9y^2).$
18. $4(6ab - 5xy)(6ab + 5xy).$
19. $(x + 19)(x + 1).$
20. $(1 - 3ab)(1 - 3ab).$
21. $(4s - 3u)(2s - 3t).$
22. $(x + y + 8)(3x + y - 8).$
23. $4ab.$
24. $(2x + y)(2x + y).$
25. $(a + 3b)(a + c).$
26. $(5a + 1)(7a + 2).$

Exercise 3. Page 10

- | | |
|--------------------|-----------------------------------|
| 1. $\frac{1}{8}.$ | 7. 1, 19. |
| 2. $\frac{1}{8}.$ | 8. 4, -3. |
| 3. $\frac{-1}{2}.$ | 9. $\frac{-3}{2}, \frac{-3}{2}.$ |
| 4. $\frac{9}{7}.$ | 10. $\frac{-2}{3}, \frac{-3}{2}.$ |
| 5. -11. | 11. $\frac{-b}{a}, \frac{-d}{c}.$ |
| 6. -12. | 12. 2, $\frac{1}{5}.$ |

Miscellaneous Problems, Chap. I. Page 10

1. (a) $\frac{a + b}{a^2 - b^2}, \frac{a - b}{a^2 - b^2};$ (b) $\frac{2a}{a^2 - b^2}.$

2. $\frac{2x-4}{x^2-7x+12}, \frac{4(x-3)}{x^2-7x+12}.$
3. (a) $\frac{(a-b)(a+b)}{(y+a)(y+a)}, \frac{2a(y+a)}{a-b};$ (b) $\frac{2a(a+b)}{y+a}.$
4. (a) $\frac{(x+3)(x+4)}{(x-2)(x-3)}, \frac{5a(x+4)}{(3x+4)(x-2)};$
(b) $\frac{(x+3)(3x+4)}{5a(x-3)}.$
6. $\frac{5}{8}.$
7. $\frac{3x+2}{4x^2-1}.$
8. $\frac{x(3a+4b)}{(9x+5)(x+3)}.$
9. $\frac{1}{abx}.$
10. $\frac{144}{55}.$
11. $\frac{1}{9(a+b)}.$
12. $\frac{x}{x-1}.$
13. $\frac{1}{64}.$
14. $\frac{a-b}{a+b}.$
15. $\frac{x(x-y)}{y(x+y)}.$
16. $\frac{-29}{12}.$
17. 1.
18. $\frac{6}{a}.$
19. $\frac{1}{6}.$
20. $\frac{x^2-x+1}{x^2}.$
21. $\frac{x^2+2xy-y^2}{x^2+y^2}.$
22. $(x+7)^2.$
23. $\frac{(x-7)^2}{(x+2)^2(x-2)}.$
24. $\frac{-2}{x(4x^2-1)}.$
25. $\frac{15-x}{60}.$
26. $\frac{a^3+b^3-c^3-ab^3}{abc}.$
27. 3.
28. -2.
29. $-7\frac{1}{16}.$
30. $x = \frac{a^2-9b}{9+b+2a}.$
31. 6, -2.
32. 7, -2.
33. $\frac{1}{2}, \frac{2}{3}.$
34. $\frac{-15}{8}, \frac{-7}{6}.$
35. $A = \frac{h(b+b')}{2}, b' = 13.6 \text{ ft.}$
36. 320 sq. in., 38.9 gal. per minute.
37. $c = 10.$
38. $\frac{d_2 n_2}{n_1} = d_1$

39. (b) 32.16 per second, per second.
 40. (a) 156 lb., approximately.
 (b) 6.2 tons, approximately.

Exercise 6. Page 30

- | | |
|---|--|
| 1. —, —, —, $C = 9n$ cts. | 4. Yes. |
| 2. A and R . | 5. Diameter and height. |
| 3. π , 4π , 9π , 16π ;
Area is a function of r . | 6. $r = \sqrt{\frac{A}{\pi}}$, $r = 6.77$. |
| | 7. —, $r = 2$. |

Exercise 7. Page 31

- | | |
|---------------------------------|----------------------------|
| 3. 15, 21, -9, $3a^2 - a - 9$. | 6. 27, 30. |
| 4. 181.6T, 70.4T, 111.2T. | 7. $a = \frac{3V}{4\pi}$. |
| 5. 79.58T. | 8. $A = 21.6$. |

Exercise 8. Page 35

- | | |
|-------------------------|---|
| 1. $1\frac{1}{8}$. | 7. $4\frac{1}{4}$ in. |
| 2. $7\frac{7}{8}$. | 8. $\frac{1}{4}$ in. = 1 lb. |
| 3. $2\frac{2}{3}$. | 9. 4.9 days. |
| 4. $2\frac{1}{4}$. | 10. $d_2 = \sqrt{2} d_1$, $A_2 = 4A_1$. |
| 5. 72. | 11. 6,000. |
| 6. $V_1: V_2 = 49:36$. | 12. $I_1 = 4I_2$. |

Supplementary Problems, Chap. I V. Page 41

3. Oats at 50 cts. per bushel = \$31.25 per ton.
 Bran at \$36 per ton is equivalent to
 (by Murry's formula):
 (a) Oats at \$37.78 per ton or 60 cts. per bushel.
 (b) Corn at \$43.80 per ton or \$1.23 per hundredweight.
 (c) Linseed meal at \$55.69 per ton or \$2.80 per hundredweight.
4. No. See parts (b) and (c) in Problem 3 and explain conclusion.
5. $\frac{S_{c.s.}}{S_{l.s.}} = \frac{n[2\frac{1}{2}(37 + 8.6) + 22]}{n[2\frac{1}{2}(30.2 + 6.7) + 33]} = 1.09:1$.
- | | |
|--------------------|-------------------|
| 6. Rye at \$30.78. | 10. 49:64. |
| 7. 0.36:1. | 11. 4,189 cu. ft. |
| 8. 14.3 acres. | 12. 1,017.9. |
| 9. 0.022 acres. | |
13. Diminished approximately 0.41 of original length.

Exercise 11. Page 47

- | | |
|--|--|
| 10. $\frac{1}{C^2}$. | 17. -1 . |
| 11. $3a^3 \cdot 3\frac{1}{2}$. | 18. $81a^8b^3b\frac{1}{2}$. |
| 12. $\frac{b^4 \cdot b\frac{1}{2}}{a^3}$. | 19. -1 . |
| 13. $b(ab)^{\frac{1}{2}}$. | 20. xy . |
| 14. $41\frac{1}{2}$. | 21. $\frac{x(x+6)}{x-6}$. |
| 15. $(a^2 + b^2)^{\frac{1}{2}}$. | 22. $\frac{(x+1)^2}{x^2(x-1)^{\frac{1}{2}}}$. |
| 16. $\frac{a^3 + 1}{a^2}$. | |

Exercise 13. Page 53

Results are given to four significant figures (for practice in interpolation) even though the data do not require it in every case.

- | | |
|------------------|---------------------------|
| 2. 0.6010. | 18. 8.640. |
| 3. 9.9943 - 10. | 19. 20.74. |
| 4. 2.7686. | 20. $6,524 \times 10^3$. |
| 5. 9.7686. | 21. 5,000. |
| 6. 4.7686 - 10. | 22. $5,434 \times 10^5$. |
| 7. 2.6706. | 23. 0.9558. |
| 8. 7.6706 - 10. | 28. 23.70. |
| 9. 9.9942 - 10. | 29. -1.329 . |
| 10. 6.9278. | 30. 0.1896. |
| 11. 4.6021. | 31. 0.08850. |
| 12. 9.9849 - 10. | 38. 0.01734. |
| 13. 4,570. | 39. 94.34. |
| 14. 20.80 | 40. 8.730. |
| 15. 100. | 43. 103.7. |
| 16. 0.9750. | 44. 52.18. |
| 17. 0.005490. | |

Exercise 14. Page 56

- | | | |
|----------|-----------|-----------|
| 1. 3.465 | 2. 3.813. | 3. 2.978. |
|----------|-----------|-----------|

Exercise 15. Page 58

- | | |
|----------|----------|
| 2. 32.8. | 3. 82.6. |
|----------|----------|

Supplementary Problems, Chap. VI Page 58

- | | |
|-------------------|---|
| 1. 12.55 minutes. | 3. $A = p(1 + r)^n$. |
| 2. 6.52 cm. | 5. $A = p\left(1 + \frac{r}{4}\right)^{4n}$. |
| 6. $A = 670.2$. | |

A larger table is necessary in order that the answer will be correct to the nearest cent.

7. $A = 672.0$. 14. 164.
 8. \$94.6. 15. 13.9 sq. cm.
 11. 57. 17. 0.007412 mm.
 13. 5.3.
 18. $2\pi r(h + r)$ = lateral area of cylinder of same height.
 19. 1,660 sq. cm.
 20. $\log y = \log 90.47 + 19.84 \log 2.056 + 208.5 \log 0.9047$
 $= 209.5 \log 9047 + 19.84 \log 2.056 - 836.0$
 $= 9.3100 - 10$
 $y = 2.042$.
 21. 47.64 per cent. 22. 25.95 per cent.

Exercise 16. Page 64

1. Each line has the slope $m = 2$. The first line has a y intercept $b = 0$, the second $b = 3$.
 2. -8, -4, 0, 8, 12.
 3. $m = 4$, $b = 0$, and $b = -6$, respectively.
 4. Yes. Yes. The equation is satisfied.
 8. 0, 0. 18. $x = -18$, $y = -28$.
 9. -2, -3. 19. $K = \frac{2}{3}$, $L = \frac{5}{3}$.
 11. $\frac{2}{13}$, $\frac{9}{13}$. 20. $x = 4$, $y = 7$.
 14. $x = -2$, $y = -1$. 21. $Q = \frac{5}{4}$, $P = 4$.
 15. $x = 1\frac{1}{3}$, $y = 1$. 22. $P = -\frac{2}{3}$.
 16. 0.7, 0.3. 23. $x = -25$, $y = -18$.
 17. $x = \frac{7}{5}$, $y = \frac{11}{5}$. 24. $x = \frac{3}{5}$, $y = \frac{1}{5}$.
 25. $x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$, $y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$.
 26. $x = 4$, $y = -\frac{1}{2}$. 29. $x = 1$, $y = 2$, $z = 3$.
 27. $x = -2$, $y = -3$, $z = 1$. 30. $x = 3$, $y = 7$, $z = 16$.
 28. $x = -2$, $y = 1$, $z = 3$.
 31. $x = \frac{b + c - a}{2}$, $y = \frac{a - b + c}{2}$, $z = \frac{a + b - c}{2}$.
 32. $x = 2$, $y = 7$, $z = 3$. 34. 4, 3.
 33. $A = \frac{1}{3}$, $B = \frac{4}{9}$, $C = \frac{2}{9}$.

Exercise 17. Page 68

2. 19.8 lb. clover hay,
 9.4 lb. corn meal.
 3. 25.4 lb. timothy hay,
 7.5 lb. cowpea meal.
 5. 21.4 lb. timothy hay,
 12.5 lb. wheat bran.
 7. (a) 18.9 lb. oats,
 5.7 lb. corn meal.

Exercise 18. Page 71

- | | |
|---|---|
| 2. 1,063 lb. dried blood,
311 lb. bone meal. | 15. 500.0 lb. nitrate,
933.3 lb. bone black,
816.3 lb. carnallit. |
|---|---|

Exercise 19. Page 75

- | | |
|------------|------------------------|
| 1. 1, -1. | 4. $\frac{1}{5}$, -6. |
| 2. -8, -1. | 5. 5, -5. |
| 3. 5, -4. | 6. -16, -9. |

Exercise 20. Page 76

- | | |
|---|---|
| 1. $\frac{-3 \pm \sqrt{69}}{10} = \text{what?}$ | 7. -2, $\frac{1}{8}$. |
| 2. 0, 7. | 8. $\frac{3}{2}$, $-\frac{1}{3}$. |
| 3. $\frac{1 \pm \sqrt{5}}{2} = \text{what?}$ | 9. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. |
| 4. $\frac{1 \pm \sqrt{289}}{3} = \text{what?}$ | 13. (a) 137.6 tons. |
| 5. 5, $\frac{1}{3}$. | 18. $d = 14$ in. |
| 6. 3, $-\frac{4}{3}$. | |

Supplementary Problems, Chap. VIII. Page 78

- | | |
|---|---|
| 1. 12 acres, approximately,
57 cows. | 6. (a) 25 ft.; (b) 23 ft. |
| 2. 26 ft.,
137 tons. | 7. $d = 2.62$, $d = -7.62$. |
| 3. 4 silos, 36 ft. | 8. $d = 42$, $d = -142$. |
| 4. 23 acres. | 9. $x = \pm \frac{1}{2}i$, $y = 1 \pm i$. |
10. (1.17, -2.66), (5.32, -5.64).
 12. $A_1 : A_2 = 1 : 4$.
 13. (a) $W_1 : W_2 = 1 : 2$;
 (b) $W_1 : W_2 = 1 : 4$;
 (c) $W_1 : W_2 = 1 : \frac{1}{2}$
 21. 11.6 in.
 22. 2 tons.
 23. 23 oz. cereal,
 33 oz. milk,
 2.2 oz. eggs.

Exercise 21. Page 84

1. $d = 2\frac{1}{3}$, $a = \frac{2}{3}$.
2. 175.
3. 500,500.
4. -180.
5. 20.
6. $L = 98$, $S = 1,010$.
7. $L = 112$, $S = 256$.

Exercise 22. Page 86

1. (a) yes; (b) yes; (c) yes.
2. 242.
3. 31.968.
4. -6,144.
5. 655.35.
6. $200(1.04)^{10} = 296.04$.
7. $p(1+r)^n$.
8. 22,960
9. $\frac{1}{90}$, 12.15 approximately.
10. $\frac{1,281}{2,560}$.
11. $n = 5$, $L = 128$.
12. seventh term = 10,240,
fifteenth term = 671,088,640.
13. $S = 4,092$.

Exercise 23. Page 87

2. 12, 192.
3. 30.
5. 18.

Exercise 24. Page 89

1. 60.
2. 156.
4. ${}_6P_6$.
5. ${}_6P_2 + {}_6P_3 = 80$.
6. ${}_6P_3 = 120$.
7. ${}_{12}P_4 = 11,880$.
8. ${}_4P_4 = 4!$

Exercise 25. Page 90

1. ${}_4C_3 = \frac{{}_4P_3}{3!} = 4$.
2. ${}_{11}C_5 = 462$.
3. (a) 3,276;
(b) 26 committees. Each committee can be formed with A and B and one of the remaining 26 professors.
4. ${}_{10}C_2 \times {}_{15}C_3$.

Exercise 31. Page 111

1. 82.
2. 6, 4.
3. $77\frac{1}{2}$.

Exercise 33. Page 113

1. 8.8 in.
2. 13 cm.
3. 148 lb. modal weight,
69 in. modal height.

Exercise 34. Page 115

1. (a) Median and mode fall in same class interval in height.
(b) Do not in weights.
(c) 69 in. median height.
(d) 143 lb. median weight.

Exercise 35. Page 119

2. \$5.0105.

3.

Frequency Table

	7	8	9	10	11	12	13	14	15	16	17	18
	1	3	5	7	18	19	18	12	3	5	6	3

Arithmetic mean 12.55 cm.

Exercise 36. Page 127

1. (a) The arithmetic mean = 3.69 ± 0.0602 ;
 (b) The standard deviation = 0.472 ± 0.043 .
2. (a) The arithmetic mean = 25.37 ± 0.11 ;
 (b) The standard deviation = 0.872 ± 0.079 .

Exercise 38. Page 134

4. (a) 0.615; (b) 0.787; (c) 0.783.

Exercise 39. Page 136

- 1.
- $\frac{3}{5}, \frac{5}{3}, \frac{4}{5}, \frac{5}{4}, \frac{3}{4}, \frac{4}{3}, \frac{3}{5}, \frac{4}{3}$
- .

Exercise 40. Page 139

1. $h = 112.8$.
2. $A = 54^\circ 50'$;
 $B = 35^\circ 10'$;
 $C = 44.21$.
3. $a = 66.51$;
 $b = 108.7$.
4. $A = 59^\circ 02'$;
 $B = 30^\circ 58'$;
 $C = 18.92$.
5. $a = 695.3$;
 $b = 923.6$.
6. $h = 98.43$.
7. $r = 28.10$.

Exercise 41. Page 140

6.	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
(a)	—	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$
(b)	$\frac{8}{17}$	—	$\frac{15}{8}$	$\frac{8}{15}$	$\frac{17}{8}$	$\frac{17}{15}$
(c)	$\frac{24}{25}$	$\frac{7}{25}$	—	$\frac{7}{24}$	$\frac{25}{7}$	$\frac{25}{24}$
(d)	$\frac{31}{39}$	$\frac{20}{39}$	$\frac{31}{20}$	—	$\frac{39}{20}$	$\frac{39}{31}$
(e)	$\frac{13}{15}$	$\frac{84}{15}$	$\frac{13}{84}$	$\frac{84}{13}$	—	$\frac{15}{13}$
(f)	$\frac{79}{87}$	$\frac{85}{87}$	$\frac{79}{85}$	$\frac{85}{79}$	$\frac{87}{79}$	—

Exercise 42. Page 142

1. (a) $\pm 45^\circ$; (b) 30° ; (c) 30° ; (d) 30° ; (e) 90° .
2. $B = 41^\circ 48'$.

Exercise 44. Page 148

2. 2,381 yd.
4. 594.9.

Exercise 45. Page 150

$$7. \cos A = \frac{2\sqrt{2}}{3}; \sin 2A = \frac{4\sqrt{2}}{9}; \cos 2A = \frac{7}{9}$$

Exercise 46. Page 153

2. 540° , 34.4° , 15° , 120° , 135° .
6. 0.0960, $\frac{3\pi}{4}$, $\frac{3\pi}{2}$, 0.0263.
5. (a) $\frac{3}{2}$; (b) 2.155.

Exercise 49. Page 163

1. $C = 65^\circ 30'$; $b = 19$; $c = 18$.
2. One solution.
3. $A = 14^\circ 47'$; $b = 803.3$; $c = 913.0$.
4. $B = 90^\circ$; $c = 468.7$.
5. $AB = 140.2$.
6. $c = 5.933$; $A = 49^\circ 15'$; $B = 108^\circ 45'$.
7. $b = 30.44$; $A = 32^\circ 56'$; $C = 117^\circ 4'$.
8. $A = 116^\circ 31'$; $B = 29^\circ 29'$; $C = 34^\circ 0'$.
10. (a) 448.4 area; (b) 33.72; (c) 282.9.

Harold P. Coleman

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Marvin Holmes.

TRANSPARENT FLEXIBLE RULER-SCALE - PROTRACTOR.

DIRECTIONS—To draw a line parallel with another, place any horizontal line on ruler directly over line and drawing edge of same is parallel thereto. To draw a line at any angle with another, place center of upper edge of ruler at point from which line is to begin; then rotate ruler until desired angle is reached. Drawing edge of ruler will then be at required angle with line. To draw circle, insert pin and pencil point in alternate holes and rotate ruler.

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